An algorithmic approach to solving polynomial equations assocciated with quantum circuits

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Main Question and Notations

Constructing unitary matrices for quantum circuit built from the Hadamard and Toffoli gates is reduced to counting a number of solutions of multivariate polynomial systems associated with the circuits.

Main Question:

How to count the number of roots for multivariate polynomial systems algorithmically and efficiently?

Notations:

$\mathbb{X} = \{x_1, \ldots, x_n\}$ is the set of polynomial variables.
$R = \mathbb{K}[\mathbb{X}]$ is a polynomial ring over field $\mathbb{K}$ of characteristic 0.
$R' = \mathbb{F}_2[\mathbb{X}]$ is a polynomial ring over field $\mathbb{F}_2 = \{0, 1\}$.
$\tilde{R} = \mathbb{F}_2[\{x_1, \ldots, x_n\} \in \mathbb{F}_2^n]$ is a polynomial ring over field $\mathbb{F}_2$, where variables belong to $\mathbb{F}_2$. Therefore multiple of monomials in this ring is defined as follow:

$$m_1 \otimes m_2 = x_1^{i_1} \cdots x_n^{i_n} \otimes x_1^{j_1} \cdots x_n^{j_n} = x_1^{\max(i_1, j_1)} \cdots x_n^{\max(i_n, j_n)}.$$

Introduction

Problem: find all the common zeros in F

$$f_1(x_1, \ldots, x_n) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_m(x_1, \ldots, x_n) = 0$$

Among the connected questions of practical interest there are:

- Is the system consistent?
- If yes, how many solutions are there?
- How to eliminate some subset of variables?
- What are the solutions?

Algorithmic methods

1. Gröbner bases (Buchberger ' 1965)
2. Involutive bases (Gerdt, Blinkov ' 1998)

Gröbner Bases application

The general strategy of the Gröbner bases approach is as follows. Given a set F of polynomials that describes the problem:

$$F = \{f_1, f_2, \ldots, f_m\}, \quad f_i \in R(R')$$

we transform F into another set G of polynomials with certain more appropriate for further research properties called a Gröbner basis, such that F and G are "equivalent" and G is "simpler" than F.

As a result:

- Because of some special properties of Gröbner basis, many problems that are difficult for general F are "easier" for Gröbner basis.
- There is an algorithm for transforming an arbitrary F into G.
- The solution of the problem for G can often be easily translated back into a solution of the problem for F.

Equivalence of G and F

Equivalence means generation of the same polynomial ideal by both sets (bases) of polynomials:

$$Ideal(F) = Ideal(G)$$

where

$$Ideal(F) := \{\sum_{i=1}^{i=m} h_i f_i \mid f_i \in F,\ h_i \in R(R')\}$$

$$Ideal(G) := \{\sum_{i=1}^{i=m} h_i g_i \mid g_i \in G,\ h_i \in R(R')\}$$

It follows, in particular, that different bases of a given ideal have the same set of roots (variety).

Gröbner bases can and were applied successfully to:

- coding theory
- cryptography
- partial differential equations
- symbolic summation and integration
- statistics
- computer geometry and graphics
- computer aided design
- numerics (e.g. wavelets and difference schemes generation)
- system theory (e.g. control theory)
- quantum computing
- .....

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Buhberger's algorithm

Algorithm: Gröbner Basis(F)

Input: $F \in R(R') \setminus \{0\}$ – finite set of polynomials
Output: G – Gröbner bases of $\mathrm{Id}(F)$

1: $B := \{[i,j] : 1 \leq i < j \leq \mathrm{length}(G)\}$
2: **while** $B \neq \emptyset$ **do**
3: $[i,j] :=$SelectPair(B, G)
4: $B := B \setminus \{[i,j]\}$
5: $h :=$NormalForm(Spoly$(G_i, G_j), G)$
6: **if** $h \neq 0$ **then**
7: $G := G \cup \{h\}$;
8: $B := B \cup \{[i, \mathrm{length}(G)] : 1 \leq i < \mathrm{length}(G)\}$
9: **end if**
10: **end while**
11: **return** G

Example

G	B
$\{x^3 - 2xy, x^2y + x - 2y^2\}$	$\{(1,2)\}$
$\{x^3 - 2xy, x^2y + x - 2y^2, -x^2\}$	$\{(1,3),(2,3)\}$
$\{x^3 - 2xy, x^2y + x - 2y^2, -x^2, -2xy\}$	$\{(2,3), (1,4),(2,4),(3,4)\}$
$\{x^3 - 2xy, x^2y + x - 2y^2, -x^2, -2xy, x - 2y^2\}$	$\{(1,4),(2,4),(3,4), (1,5),(2,5),(3,5),(4,5)\}$

10 reductions, 7 unnecessary reductions

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Involutive algorithm

Algorithm: JanetBasis(F)

choose $f \in F$ such that $\mathrm{lm}(f) = \min\{\mathrm{lm}(F)\}$
$G := \{f\}$
$Q := F \setminus G$
while $Q \neq \emptyset$ **do**
    choose $p \in Q$ such that $\mathrm{lm}(p) = \min\{\mathrm{lm}(Q)\}$
    $Q := Q \setminus \{p\}$
    $h :=$JanetNormalForm(p, G)
    if $h \neq 0$ **then**
        $G := G \cup \{h\}$
        for all $q \in G$ and $x \in NM_J(q, G)$ **do**
            $Q := Q \cup \{q \cdot x\}$
        end for
    end if
end while
return G

Example

G	Q
$\{x^2y + x - 2y^2\}$	$\{x^3 - 2xy\}$
$\{x^2y + x - 2y^2, x^3 - 2xy\}$	$\{x^3y + x^2 - 2xy^2\}$
$\{x^2y + x - 2y^2, x^3 - 2xy, x^2\}$	$\{x^3, x^2y\}$
$\{x^2y + x - 2y^2, x^3 - 2xy, x^2, 2xy\}$	$\{x^2y, 2x^2y\}$
$\{x^2y + x - 2y^2, x^3 - 2xy, x^2, 2xy, -x + 2y^2\}$	$\{2x^2y, -x^2 + 2xy^2, -xy + 2y^3\}$
$\{x^2y + x - 2y^2, x^3 - 2xy, x^2, 2xy, -x + 2y^2\}$	$\{-x^2 + 2xy^2, -xy + 2y^3\}$
$\{x^2y + x - 2y^2, x^3 - 2xy, x^2, 2xy, -x + 2y^2\}$	$\{-xy + 2y^3\}$
$\{x^2y + x - 2y^2, x^3 - 2xy, x^2, 2xy, -x + 2y^2, 2y^3\}$	$\{\}$

6 reductions, 2 unnecessary reductions

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Ring $\tilde{R}$

Every monomial order in $\tilde{R}$ is not an admissible one:

$$m_1 = x_1^{i_1} \cdots x_n^{i_n},\ m_2 = x_1^{j_1} \cdots x_n^{j_n},\ m_1 \neq m_2 \Longrightarrow m_1 \prec m_2 \text{ or } m_1 \succ m_2$$

$$m_3 = \operatorname{lcm}(m_1, m_2) = x_1^{\max(i_1, j_1)} \cdots x_n^{\max(i_n, j_n)}$$

$$m_1 \cdot m_3 = x_1^{\max(i_1, \max(i_1, j_1))} \cdots x_n^{\max(i_n, \max(i_n, j_n))} = m_3$$

$$m_2 \cdot m_3 = x_1^{\max(j_1, \max(i_1, j_1))} \cdots x_n^{\max(j_n, \max(i_n, j_n))} = m_3$$

A single polynomial basis in $\tilde{R}$ is not a Gröbner basis for sure.
Example: $\langle xy + x + 1 \rangle = \langle x + 1, y \rangle$.

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Ideals in $\tilde{R}$ and R'

Theorem

Let us have $\{f_1, \ldots f_m\}$ – a finite set of polynomials in $\tilde{R}$. Using homomorphism from R' to $\tilde{R}$ a basis $\{\tilde{f}_1, \ldots, \tilde{f}_m, B\}$, where $\tilde{f}_i$ is the original of f_i in R' and $B = \{x_1^2 + x_1, \ldots, x_n^2 + x_n\}$, can be obtained. Applying any algorithm gives us a Gröbner basis – $\{\tilde{g}_1, \ldots, \tilde{g}_k, B'\}$, $B' \subseteq B$. Then the set $\{g_1, \ldots, g_k\}$ (g_i – image of $\tilde{g}_i$) is the required Gröbner basis of ideal $\langle f_1, \ldots, f_m \rangle$ in the ring $\tilde{R}$.

Therefore:
from now on we use the ring R' instead of $\tilde{R}$ for constracting desired Gröbner bases.

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Vectorization

1-bit vectorization:

$$x_0 x_1 x_4 x_5 = \underbrace{\langle 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \rangle}_{64(128)\,bits}$$

2-bit vectorization:

$$x_0 x_1^2 x_4 x_5 = \underbrace{\langle 00\ 00\ 00\ 01\ 01\ 00\ 10\ 01 \rangle}_{64(128)\,bits}$$

Benchmarks

Used software and hardware

CoCoA 4.6, Singular 3.0.2, Mathematica 5.0,
GBF2 0.2.3 Buhberger and GBF2 0.3.5 Involutive.
Machine: 2xOpteron-242 (1.6 Ghz) with 6Gb RAM under Gentoo Linux 2005.1, gcc-4.1.0 compiler.

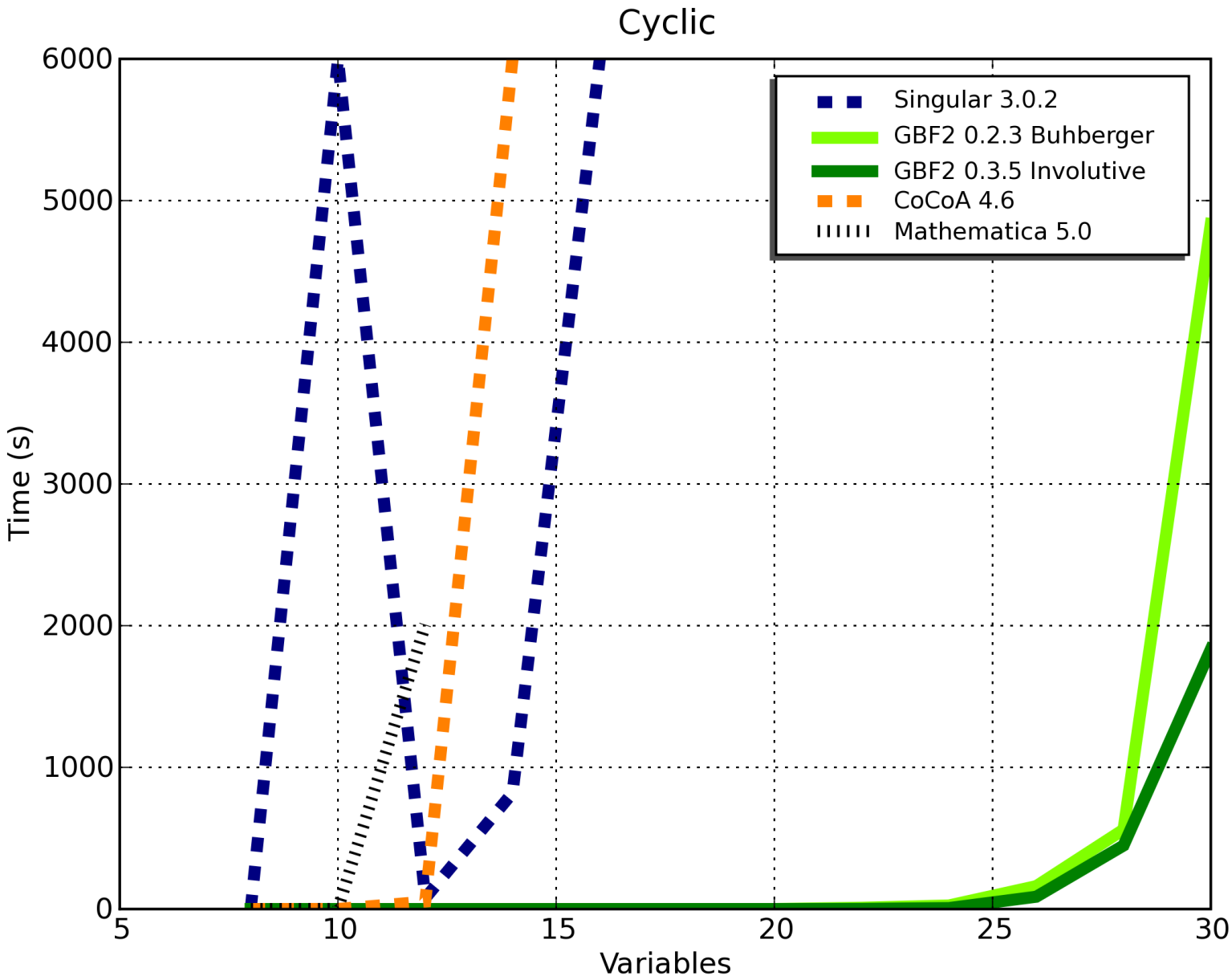
Used series

Cyclic, Eco, Redcyc, Redeco, Noon, Katsura – stantard series (http://www-sop.inria.fr/saga/POL).
Series Life: a single polynomial of variables $x_0, \ldots, x_i$

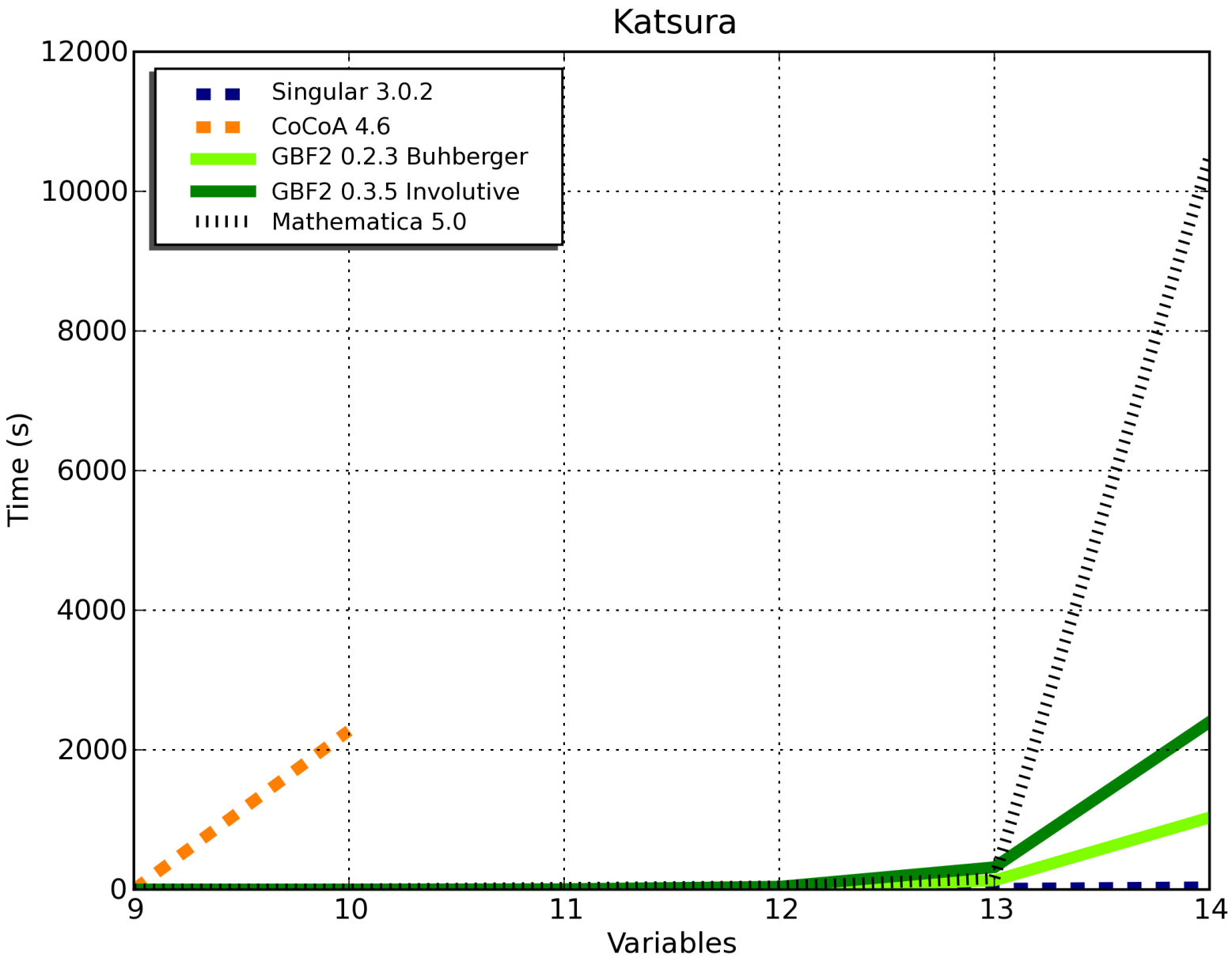
$$x_i + x_{i-1}(\sigma_{i-2} + \sigma_{i-3} + \sigma_3 + \sigma_2) + \sigma_{i-2} + \sigma_3$$

where σ_k is the $k-$th symmetric polynomial of variables $x_0, \ldots, x_{i-2}$.

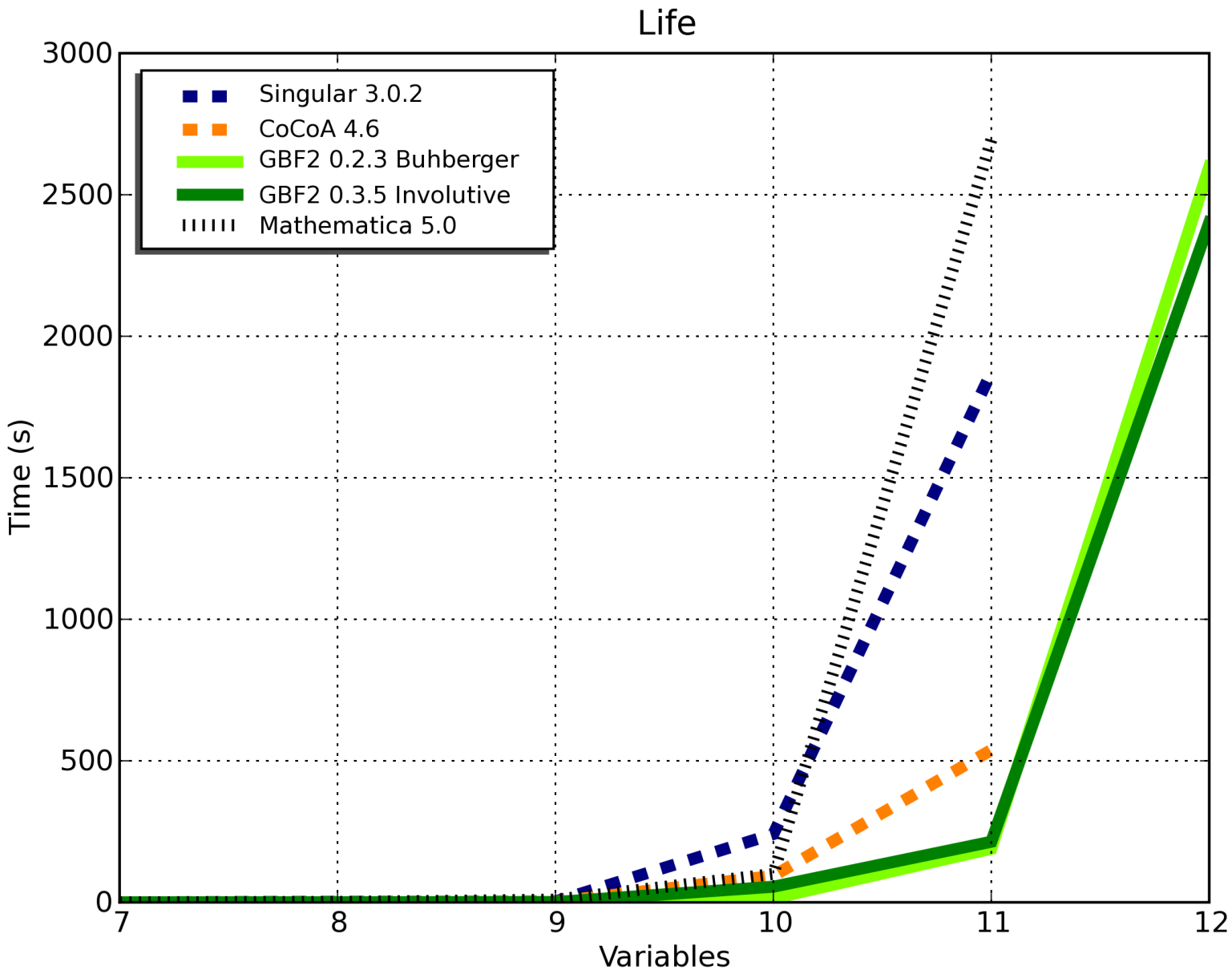
Cyclic series



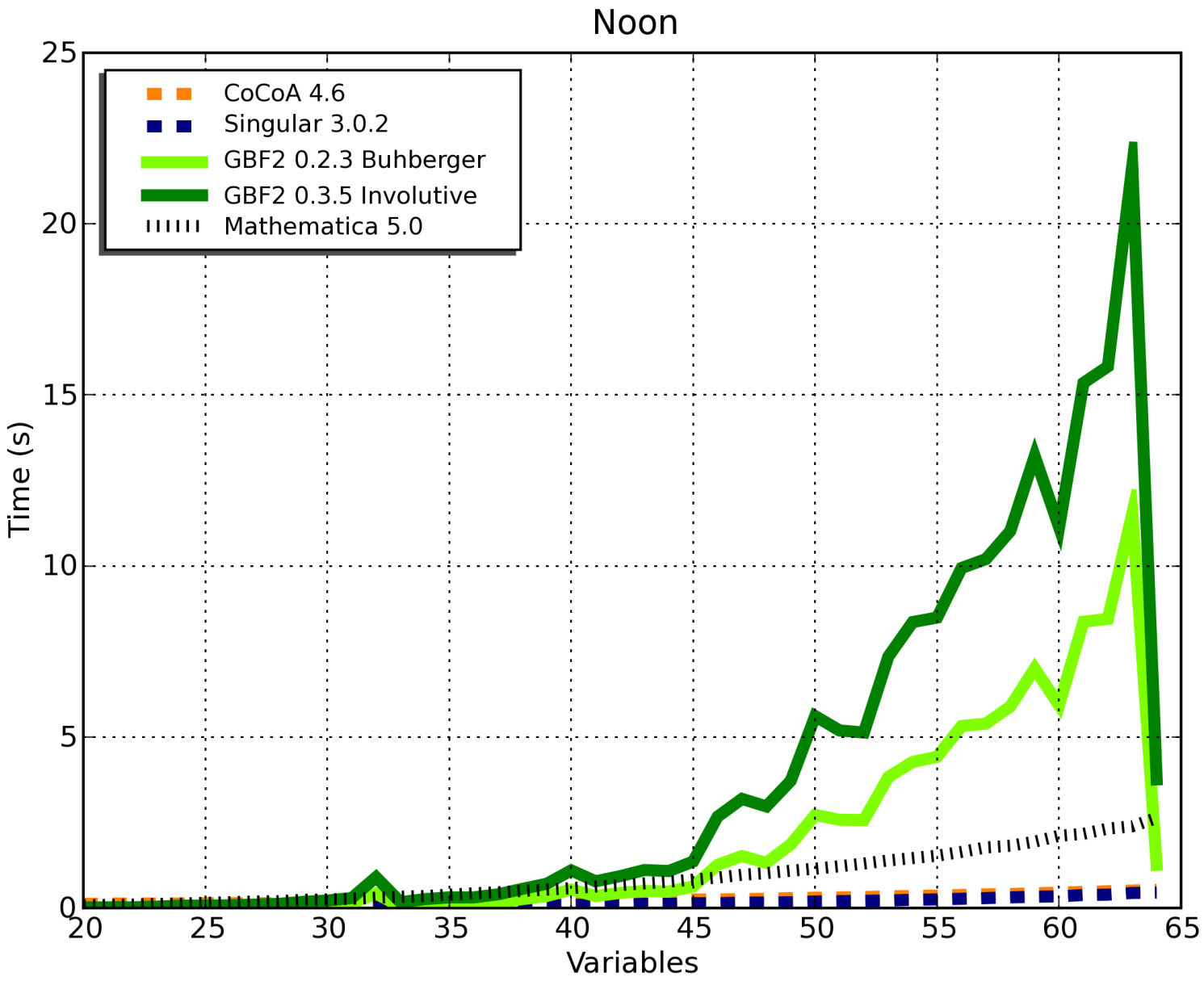
Katsura series



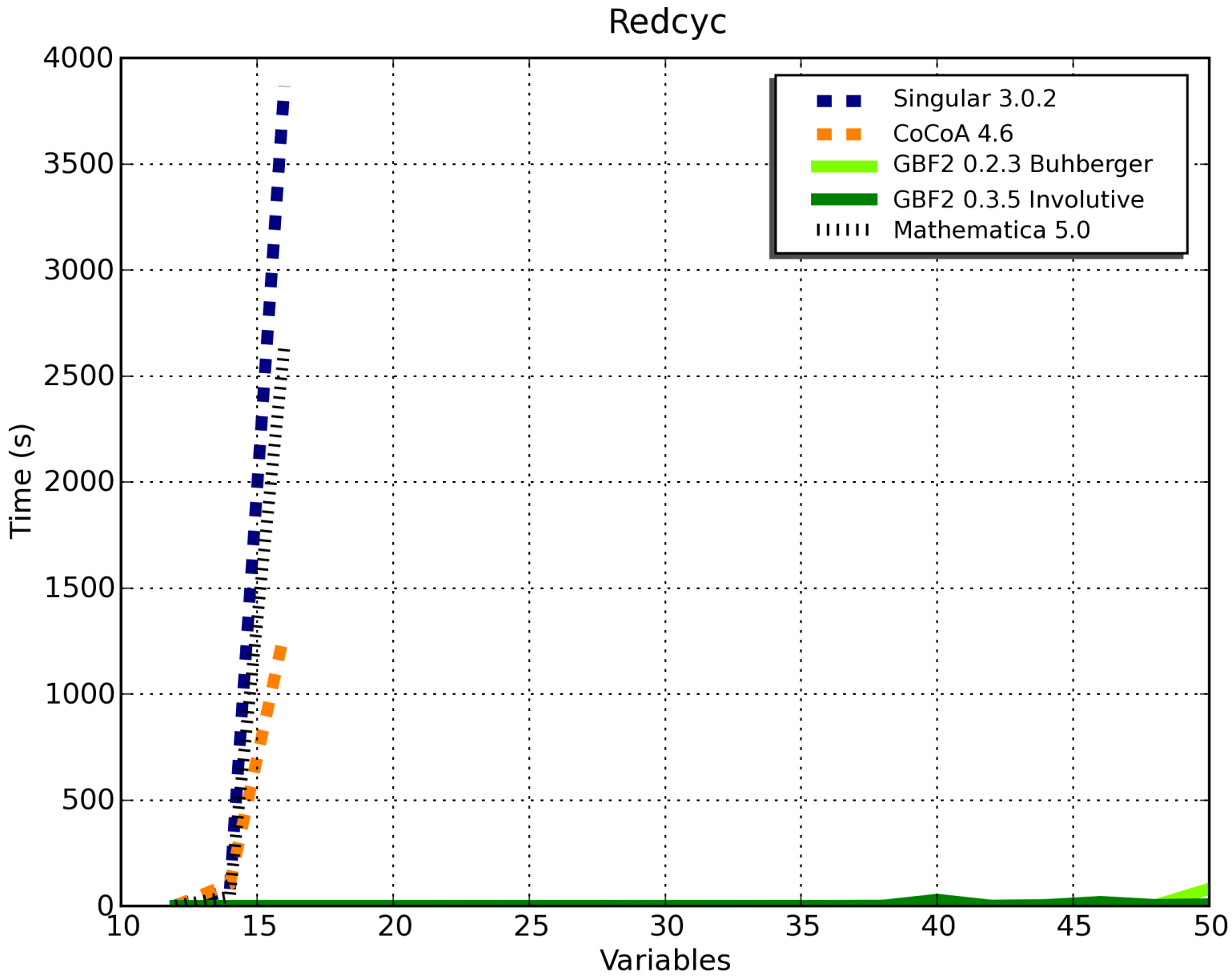
Life series


Life
Singular 3.0.2
CoCoA 4.6
GBF2 0.2.3 Buhberger
GBF2 0.3.5 Involutive
Mathematica 5.0
Time (s)
Variables
0
500
1000
1500
2000
2500
3000
7
8
9
10
11
12

Noon series


Noon
CoCoA 4.6
Singular 3.0.2
GBF2 0.2.3 Buhberger
GBF2 0.3.5 Involutive
Mathematica 5.0
Time (s)
Variables
0
5
10
15
20
25
20
25
30
35
40
45
50
55
60
65

Redcyc series



Further developments

- Improvement of inner data structures (1-bit vectorization, etc.)
- Search for more appropriate strategy for selection of
  1. critical pairs (Buchberger's algorithm)
  2. nonmultiplicative prolongations (Involutive algorithm)
- Implementation of the FGLM algorithm for converting of degree-reverse-lexicographical Gröbner or involutive bases (that are computed much faster then their pure lexicographical counterparts) into the corresponding lexicographical bases.
- Using division other that Janet one (for example Pommaret, etc.) and more appropriate for computation over $\mathbb{F}_2$
- Installation as a program module in the specialized computer algebra system GINV (see http://invo.jinr.ru).