Numerical Study of Time-periodic Solitons in the Damped-driven Nonlinear Schrödinger Equation

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In the frame of the scientific collaboration between UCT (Cape Town) and JINR (Dubna), we study spatially localized solutions of the parametrically driven damped nonlinear Schrödinger equation,

\[ i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma \psi, \]  

(1)

where \( \gamma > 0 \) is the damping coefficient, \( h > 0 \) the amplitude of the parametric driver, and symbol "*" means the complex conjugation.

Equation (1) describes small and slowly varying amplitudes of waves and patterns in spatially distributed parametrically driven systems. It was employed to model intrinsic localized modes in coupled microelectromechanical and nanoelectromechanical resonators, solitons in dual-core nonlinear optical fibers, and dissipative structures in optical parametric oscillators. More applications of (1) are listed in [1, 2].

Equation (1) exhibits different classes of soliton solutions existing on the \( (h, \gamma) \)-plane above the straight line \( h = \gamma \). Two stationary solitons \( \psi_+ \) and \( \psi_- \) are known explicitly. Stability properties of \( \psi_+ \) and \( \psi_- \) have been investigated in [3]. Stable stationary solutions in the form of association of two solitons, are obtained, numerically, in [4].

In [1, 2, 5, 6, 7, 8] we study temporally periodic, spatially localized solutions of (1) that arise as the Hopf bifurcation (HB) of stationary solitons.

The standard way to obtain periodic solutions is the direct numerical simulation. The shortcoming of this method is that simulations capture only stable regimes. This means that the actual mechanisms and details of the transformations (which are bifurcations involving both stable and unstable solutions) remain unaccessible. Neither can simulations be used to identify alternative attractors in cases of bistability or multistability.

We propose a new approach to the analysis of these hidden mechanisms. Instead of direct numerical simulations, the time-periodic solitons are studied as solutions to a boundary-value problem formulated on a two-dimensional cylinder. The advantage of this approach is that it furnishes both stable and unstable solutions.

We are looking for time-periodic solutions by solving (1) as a boundary-value problem on a two-dimensional domain \( (-\infty, \infty) \times (0, T) \). The boundary conditions are set as

\[ \psi(x, t) = 0 \quad \text{as } x \to \pm \infty, \]  

(2)

and

\[ \psi(x, t + T) = \psi(x, t). \]  

(3)

The period \( T \) is regarded as an unknown, along with the solution \( \psi(x, t) \).

Letting \( \bar{t} = t/T (0 < \bar{t} < 1) \) and defining \( \tilde{\psi}(x, \bar{t}) = \psi(x, t) \), the boundary-value problem (1), (2), (3) can be reformulated on the rectangle \( (-L, L) \times (0, 1) \) (where \( L \) is chosen to be sufficiently large):

\[ F \equiv \bar{\psi}_{\bar{t}}(x, \bar{t}) + T\Phi(\tilde{\psi}(x, \bar{t}), h, \gamma) = 0, \]  

(4)

\[ \tilde{\psi}(\pm L, \bar{t}) = 0, \]  

(5)

\[ \tilde{\psi}(x, 0) = \psi(x, 1). \]  

(6)

Here,

\[ \Phi(\tilde{\psi}(x, \bar{t}), h, \gamma) = \tilde{\psi}_{xx} + 2|\tilde{\psi}|^2\tilde{\psi} - \tilde{\psi} - h\tilde{\psi}^* + i\gamma \tilde{\psi}. \]  

(7)

Equations (4-6) are supplemented with an additional equation in the form [9]:

\[ \Phi(\tilde{\psi}(x^*, \bar{t}^*), h, \gamma) = 0, \quad x^* = t^* = 0. \]  

(8)

Solutions \((T, \tilde{\psi})\) of the 2D boundary-value problem (4-8) were path-followed in \( h \) for the fixed \( \gamma \), with the Hopf bifurcation points (see Fig. 1) of the static solution used as starting points in the continuation process. We employ a predictor-corrector algorithm [10] with iteration process at each \( h \) on the basis of continuous analog of Newton’s method [11]. Details of our numerical approach are given in [7].

Stability and bifurcations of solutions are classified by examining the Floquet multipliers of the corresponding linearised equation. Details are in [1, 5].

Representative time-periodic single- and two-solitons are demonstrated in Fig. 2.

All time-periodic single-soliton solutions are found to be symmetric in space. Periodic complexes starting from the HBs 1, 2, 3 (see Fig. 1) are also symmetric. However, the fourth HB gives a born to a complex of two solitons with out-of-phase oscillations in time, see Fig. 3.

Periodic branches associated with HBs 2, 3 are established to lose stability at the points of Neimark-Sacker bifurcation where the branches of quasiperiodic solutions are born. Quasiperiodic solutions can
Figure 1: (Color online) Stability domain of stationary $\psi(+)$ and $\psi(++)$. Single-soliton solutions exist for $h$ between the curve $h_{cont}$ and the straight line $h = \gamma$. Stability domain of the $\psi_+$ soliton is bounded by the dashed curve $h_{Hopf}$, solid curve $h_{cont}$ and the straight line $h = \gamma$ [3]. The complex $\psi_{(++)}$ exists between the dotted line and the curve $h_{cont}$. The complex is stable in a tinted part of this region. The stability domain is bounded by the curve $h = h_{cont}$ on the top and by a lines of the Hopf bifurcations (numbered 1, 2, 3, 4) on other sides.

not be obtained in the frame of numerical approach presented above. They were obtained in direct numerical simulation of (1). We employed the pseudospectral method [12] based on the Fourier discretization in space and numerical solution of the resulting initial-value ODE problem with help of the Runge-Kutta algorithm. On the basis of direct simulations with different values of $h$ and $\gamma$ we established a stability region of quasiperiodic complex.

Figure 4 summarizes our conclusions on single-soliton periodic attractors. This diagram is in good agreement with the attractor chart produced using direct numerical simulations [13].

The diagram on Fig. 5 complements the single-soliton attractor chart with the stationary, periodic, and quasiperiodic two-soliton attractors.

Fig. 5 reproduces the stability region of stationary complexes shown above in Fig. 1. Stability domain of time-periodic complexes consists of stability regions of complexes emerging in HBs 1, 2, 3, 4. A schematic position of the quasiperiodic stability domain is also given as result of direct numerical simulations.

Stationary and time-periodic solitons are shown to coexist with stationary, time-periodic and quasiperiodic two-soliton solutions. Note for example, for $\gamma = 0.35$, the periodic freestanding soliton and periodic two-soliton complex coexist between $h = 0.806$ and $h = 0.832$. However the time-periodic one- and two-soliton branches are not connected.

Final observation concerns the coexistence of stationary and periodic complexes. At this stage, we only marked a small portion of this bistability domain (indicated by the black marker on Fig. 5).

Note in conclusion, that our numerical approach requires substantial computational resources (computer memory and processor time) to perform computations with a reasonable accuracy. The computationally inexpensive approximate method has been proposed in [8]. The partial differential equation (PDE) is reduced to a system of coupled ordinary differential equations, which is then solved numerically, on a one-dimensional domain. The reduced system allows not only determining the domain of existence of the breathers but also studying their stability and bifurcations.
Figure 3: (Color online) Stable periodic two-soliton complex oscillating out of phase with each other. Here $h = 1.0493$ and $T = 1.991$. Several periods of oscillation are shown.

Figure 4: (Color online) Single-soliton attractor chart

References


