Super Energy Tensors in Bianchi Type I Universe

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Bel-Robinson tensor (BR) first appeared in the endless search for a covariant version of gravitational energy; the analogy with the Maxwell stress tensor $T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} + *F_{\mu\alpha} * F_{\nu}^{\alpha}$. In general BR is a 4-rank tensor with the following symmetry properties

$$
\begin{aligned}
B_{\mu\nu\alpha\beta} &= B_{\nu\mu\alpha\beta}, && \text{(1a)}\\
B_{\mu\nu\alpha\beta} &= B_{\mu\nu\beta\alpha}, && \text{(1b)}\\
B_{\mu\nu\alpha\beta} &= B_{\alpha\beta\mu\nu}. && \text{(1c)}
\end{aligned}
$$

In literature there are a few definitions of BR. Here we mention only three.

I. Deser *et. al.* defined BR as

$$
B_{\mu\nu\alpha\beta} = R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} R_{\rho\nu\sigma\beta} + *R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} * R_{\rho\nu\sigma\beta}, \tag{2}
$$

Here the dual curvature is $*R^{\mu\nu}_{\ \ \ \ \lambda\sigma} \equiv (1/2)\,\epsilon^{\mu\nu}_{\ \ \ \ \alpha\beta} R^{\alpha\beta}_{\ \ \ \ \lambda\sigma}$. Using the properties of totally anti-symmetric Levi-Civita tensor $\epsilon_{\alpha\beta\mu\nu}$ definition (2) can be written alternatively as

$$
B_{\mu\nu\alpha\beta} = R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} R_{\rho\nu\sigma\beta} + R^{\rho\ \ \sigma}_{\ \mu\ \ \beta} R_{\rho\nu\sigma\alpha} - \frac{1}{2} g_{\mu\nu} R_{\alpha}^{\ \ \rho\sigma\tau} R_{\beta\rho\sigma\tau}. \tag{3}
$$

The properties (1a) and (1b) follow immediately from (2) thanks to the symmetry property of Riemann tensor. The property (1c) is strait forward from (2), but for (3) it requires

$$
g_{\mu\nu} R_{\alpha}^{\ \ \rho\sigma\tau} R_{\beta\rho\sigma\tau} = g_{\alpha\beta} R_{\mu}^{\ \ \rho\sigma\tau} R_{\nu\rho\sigma\tau}. \tag{4}
$$

Thus, defined as (2) or (3) BR imposes following restrictions on the metric functions of Bianchi type I metric:

$$
\frac{\ddot a}{a} = \frac{\dot b}{b}\frac{\dot c}{c}, \quad \frac{\ddot b}{b} = \frac{\dot c}{c}\frac{\dot a}{a}, \quad \frac{\ddot c}{c} = \frac{\dot a}{a}\frac{\dot b}{b}. \tag{5}
$$

II. Teyssandier and many others define BR as

$$
2B_{\mu\nu\alpha\beta} = R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} R_{\rho\nu\sigma\beta} + *R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} * R_{\rho\nu\sigma\beta} + R*^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} R*_{\rho\nu\sigma\beta} + *R*^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} *R*_{\rho\nu\sigma\beta}, \tag{6}
$$

where the duality operator acts on the left or on the right pair of indices according to its position. From (6) one easily finds

$$
\begin{aligned}
B_{\mu\nu\alpha\beta} &= R^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} R_{\rho\nu\sigma\beta} + R^{\rho\ \ \sigma}_{\ \mu\ \ \beta} R_{\rho\nu\sigma\alpha} - \frac{1}{2} g_{\mu\nu} R_{\alpha}^{\ \ \rho\sigma\tau} R_{\beta\rho\sigma\tau} \\
&- \frac{1}{2} g_{\alpha\beta} R_{\mu}^{\ \ \rho\sigma\tau} R_{\nu\rho\sigma\tau} + \frac{1}{8} g_{\mu\nu} g_{\alpha\beta} R^{\rho\sigma\tau\eta} R_{\rho\sigma\tau\eta}.
\end{aligned} \tag{7}
$$

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Under the new definition the symmetry properties (1a),(1b) and (1c) follow immediately, without any restriction to the metric functions. But in this case BR can be totally symmetric if and only if $R_{\mu\nu} = 0$.

III. Finally, Senovilla, Berqgvist and others define BR using well known Weyl tensor $C_{\alpha\beta,\mu\nu}$

$$B_{\mu\nu\alpha\beta} = C^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} C_{\rho\nu\sigma\beta} + {*C}^{\rho\ \ \sigma}_{\ \mu\ \ \alpha} * C_{\rho\nu\sigma\beta}. \tag{8}$$

In this case the properties (1) fulfills. Moreover, BR in this case becomes trace-free, i.e.,

$$g^{\mu\nu} B_{\mu\alpha\nu\beta} = 0. \tag{9}$$

So defined, the BR also possesses dominant property:

$$B_{\mu\alpha\nu\beta} u^\mu u^\alpha u^\nu u^\beta \geq 0, \tag{10}$$

with u^σ being some causal future-pointing vectors.

Within the framework of Bianchi type-I (BI) space-time we study the Bel-Robinson tensor and its impact on the evolution of the Universe. We use different definitions of the Bel-Robinson tensor existing in the literature and compare the results [1]. The dominant property (DP) has been investigated and it is shown that at least in case of Kasner universe, which is a partial case of BI space-time, DP holds.

References

[1] Bijan Saha, Victor Rikhvitsky and Mihai Visinescu: *Bel-Robinson tensor for the Bianchi type I universe* Romanian Report of Physics **57** (4) (2005) (to be published).