Cohomology of Restricted Algebras of Hamiltonian Vector Fields

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The restricted Lie algebras are defined over fields of characteristic p. We present here three statements valid for all restricted Lie algebras of vector fields with the Poisson brackets. These statements were first revealed directly by computer with the help of author's program [1] and then proved rigorously [2, 3]. We present also the results of full computation of cohomology for the restricted Lie algebra of 2-dimensional Hamiltonian vector fields in the characteristic 5.

Traditionally when computing cohomology of Lie (super)algebras, mathematicians use the *standard* (called also *Weisfeiler's*) grading. We introduce another type of grading — we call it *symmetric* and show that in the case of restricted Lie algebras the symmetric grading can be used for computing cohomology. Moreover, the symmetric grading provides all Poisson and Hamiltonian algebras with the *inner grading element* having remarkable mathematical properties.

For illustration in Table 1 we compare both gradings presenting dimensions of cohomologies for some small restricted Hamiltonian algebra.

Table 1: $\dim H^k_g\left(\mathfrak{h}(2)_3\right)$

Standard (Weisfeiler's) grading

$g\backslash k$	0	1	2	3	4	5	6	7	8	9	10
-2			1	·	·	1					
-1		·	·	·	·	·	·				
0	1	·	·	4	1	·	3	1			
1		2	·	·	4	2	·	2	2		
2		·	·	·	·	·	·	·	·	·	
3			2	2	·	2	4	·	·	2	
4				1	3	·	1	4	·	·	1
5					·	·	·	·	·	·	
6						1	·	·	1		

Symmetric grading

$g\backslash k$	0	1	2	3	4	5	6	7	8	9	10
-7					·	·	·				
-6				1	1	·	1	1			
-5			·	·	·	·	·	·	·		
-4			·	·	·	·	·	·	·		
-3		1	1	1	2	2	2	1	1	1	
-2		·	·	·	·	·	·	·	·	·	
-1		·	·	·	·	·	·	·	·	·	
0	1	·	1	3	2	2	2	3	1	·	1
1		·	·	·	·	·	·	·	·	·	
2		·	·	·	·	·	·	·	·	·	
3		1	1	1	2	2	2	1	1	1	
4			·	·	·	·	·	·	·		
5			·	·	·	·	·	·	·		
6				1	1	·	1	1			
7					·	·	·				

Looking at the right hand part of this table we can immediately formulate and prove the following propositions. Below $\mathfrak{a}(n)_p$ denotes any restricted Hamiltonian or Poisson algebra, n is dimension of the symplectic manifold, p is characteristic of the base field.

Proposition 1. *Cohomologies in subcomplexes with opposite grades are isomorphic:*

$$H_g^k\left(\mathfrak{a}(n)_p\right) \cong H_{-g}^k\left(\mathfrak{a}(n)_p\right) .$$

Proposition 2. *Cohomology of degree k in a given grade g is isomorphic to the homology of degree $N-k$ in the same grade g* (Poincaré duality)*:*

$$H_g^k\left(\mathfrak{a}(n)_p\right) \cong \left(H_g^{N-k}\left(\mathfrak{a}(n)_p\right)\right)' = H_{N-k,g}(\mathfrak{a}(n)_p) .$$

Proposition 3. *For the symmetric grading all non-trivial cohomologies $H_g^k\left(\mathfrak{a}(n)_p\right)$ lie in the grades g which are multiple of the characteristic p: $g \equiv 0 \mod p$.*

These propositions essentially reduce difficulties in computation of cohomologies. In Table 2 we present the results of full computation of cohomology for the Lie algebra $\mathfrak{h}^{(2)}(2)_5$. Note that this is rather difficult task since the total dimension of cochain space is equal here to 8388608. The computation has been performed on a 1133MHz Pentium III PC with 512Mb.

Table 2: $\dim H_g^k\left(\mathfrak{h}^{(2)}(2)_5\right)$.

$g \setminus k$	0 23	1 22	2 21	3 20	4 19	5 18	6 17	7 16	8 15	9 14	10 13	11 12
0	1	·	1	1	4	4	8	12	9	18	14	30
± 5			1	1	3	2	6	9	8	15	14	25
±10				·	3	1	3	6	4	9	7	17
±15							2	1	1	4	3	7
±20											1	3

References

[1] *Kornyak V.V.* Modular Algorithm for Computing Cohomology: Lie Superalgebra of Special Vector Fields on (2|2)-dimensional Odd-Symplectic Superspace. In: "Computer Algebra in Scientific Computing / CASC 2003", V.G.Ganzha, E.W.Mayr, E.V.Vorozhtsov (Eds.). Institute of Informatics, Technical University of Munich, München, 2003, 227–240; id., http://arXiv.org/abs/math.RT/0305155.

[2] *Kornyak V.V.* On the Structure of Cohomology of Hamiltonian p-Algebras. In: "Computer Algebra in Scientific Computing / CASC 2004", V.G.Ganzha, E.W.Mayr, E.V.Vorozhtsov (Eds.). Institute of Informatics, Technical University of Munich, München, 2004, 307–311; id., http://arXiv.org/abs/math.RT/0404247.

[3] *Kornyak V.V.* Cohomologies of Restricted Lie Algebras of Hamiltonian Vector Fields: Computer Analysis. Russian Journal for Computer Science ("Programmirovanie"), Vol.31, No.2, 2005, 87–90 (in Russian), Programming and Computer Software, Vol.31, No.2, 2005, 87–90 (English translation).