Discrete Relations on Abstract Simplicial Complexes

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We propose a new universal mathematical frame for constructing models in mathematical physics. This construction — we call it *a system of discrete relations on an abstract simplicial complex* — can be interpreted as

- a natural generalization of the notion of cellular automaton:
  - instead of regular uniform lattice representing the space and time in a cellular automaton, we consider a more general *abstract simplicial complex*. One of the advantages of simplicial complexes over regular lattices is their applicability to models with dynamically emerging and evolving rather than a pre-existing space-time structure.
  - instead of the local site-update function of a cellular automaton we consider a much more flexible set of relations imposed on simplices of the complex. In this context, the local site-update function is a special case of the relation — the so-called *functional relation*.
- a set-theoretic analog of a system of polynomial equations:
  - if the number of states q is a power of a prime, i.e., $q = p^n$, we can always represent any relation over k points $\{x_1, \ldots, x_k\}$ by the set of zeros of some polynomial from the ring $\mathbb{F}_q[x_1, \ldots, x_k]$. ($\mathbb{F}_q$ is a Galois field.)

After introducing appropriate definitions *(extension of relation, consequence of relation, proper consequence, prime relation, base relation* $\equiv$ *compatibility condition, canonical decomposition, principal factor, reducible relation* etc.), we develope and implement algorithms for

- *compatibility analysis* of a system of discrete relations;
- constructing *canonical decompositions* of discrete relations.

We propose a regular way to impose *topology on an arbitrary discrete relation* via its canonical decomposition: identifying *dimensions* of the relation with *points* and *irreducible components* of the relation with *maximal simplices*, we define the structure of an abstract simplicial complex on the relation under consideration. Now we can evolve — starting only with a set of points and a relation on it — the standard tools of the algebraic topology like homology group, cohomology ring, etc.

Applying the above technique to cellular automata — a special case of a system of discrete relations — we have obtained some new results. Most interesting of them, in our opinion, is observation that the presence of *non-trivial proper consequences* may determine the global behavior of an automaton.

To demonstrate this, let us look at a typical pattern from Wolfram's online atlas http://atlas.wolfram.com/01/01/. In Fig. 1 several evolutions of the elementary cellular automaton 168 are presented. The black and white square cells in Fig. 1 correspond to 1's and 0's, respectively.

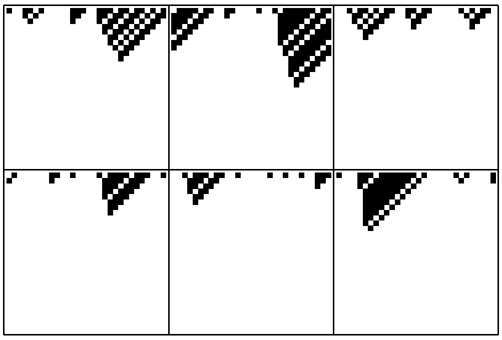
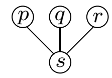
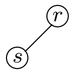


Figure 1: Rule 168. Several random initial conditions

We see that all patterns are unions of down-left-directed strips either containing only 0's or beginning with several initial 1's followed by 0's.

This peculiarity in the behavior of the automaton is explained by the fact that the local relation of the automaton — defined on the simplex 

and taking the form $pqr + qr + pr + s = 0$ in terms of polynomial over $\mathbb{F}_2$ — has the proper consequence $rs + s = 0$ on the face 

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The results mentioned in this note are published in papers [1, 2].

References

[1] *Kornyak V.V.* On Compatibility of Discrete Relations. Lecture Notes in Comp. Sci. **3718**, Springer-Verlag, 2005, 272–284; id., http://arXiv.org/abs/math-ph/0504048.

[2] *Kornyak V.V.* Discrete Relations On Abstract Simplicial Complexes. Russian Journal for Computer Science ("Programmirovanie"), (in Russian, to be published).