Inverse Problem for Discrete Elliptic Equation with Prescribed Symmetry Conditions

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Multidimensional inverse problems are very interesting for applications to nuclear physics, geophysics, tomography and others [1]. Many physical processes are described by PDE which can be solved numerically only. We developed [2],[3] an original numerical-analytical algorithm for reconstructing the two-dimensional discrete elliptic equation:

$$-(\kappa^1_{i+1/2,j}(u_{i+1,j}-u_{i,j})-\kappa^1_{i-1/2,j}(u_{i,j}-u_{i-1,j}))h_x^{-2}$$

$$-(\kappa^2_{i,j+1/2}(u_{i,j+1}-u_{i,j})-\kappa^2_{i,j-1/2}(u_{i,j}-u_{i,j-1}))h_y^{-2}+p_{i,j}u_{i,j}=\lambda u_{i,j}.$$

The right problem is solved in rectangle $1 \le i \le M,\ 1 \le j \le N$, with zero boundary conditions $u_{0,j}=u_{M+1,j}=u_{i,0}=u_{i,N+1}=0$. This problem can appear in a numerical simulation of wave processes in periodic media [4],[5]. The prescribed symmetry conditions give a possibility to prolong a discrete eigenfunction from the rectengular to the whole plain reserving "continuity of the first derivatives" which is important for applications. The problem is reduced to recostruction of a block three-diagonal matrix. We prove the Symmetry Theorem said that with the prescribed symmetry conditions the found matrix and its blocks are symmetric with respect to both diagonals (persymmetric matrices [6]). We proved in addition that the basic eigenvectors of the found matrix reserve all symmetries of original matrix, corresponding to $p_{i,j}=0$ and $\kappa^1_{i\pm1/2,j}=\kappa^2_{i,j\pm1/2}=1$. As a result, the spectrum of the found matrix splits in 4 nonintersect sets $Spij,\ 1 \le i,j \le 2$. We derive clear views for polynomials equations $Pij(\lambda)=0$, determining elements of $Spij$. The developed numerical-analytical algorithm is the following: the corresponding persymmetric matrix has $L < MN$ different elements and can be reconstructed of L given eigenvalues. The matrix elements and $MN-L$ lacking eigenvalues can be found by solving a polynomial system, constituted of the Vieta relations for the polynomials $Pij(\lambda)$. We developed as well an algorithm for the basic vectors calculation. We present one of produced numerical experiments: $(M=5,\ N=6)$, the persymmetric five-diagonal matrix of order 30 was reconstructed. 24 lowest original eigenvalues were disturbed: in every case 2 first signes after the point were reserved and the rest were cut off. 24 different matrix elements and 6 lacking eigenvalues were found by solving the system of 30 polynomial equations being the Vieta relations for coefficients of the polynomials P_{11}, P_{12}, P_{21}, P_{22}. The first 2 have order 9 of λ, and the other 2 have order 6. The cumbersum system of 30 polynomial equations was calculated by using CAS REDUCE [7]. This system was solved by implemented the NUMERIC package by G.Melenk [7]. The coefficients of the found discrete elliptic equations $\kappa^1_{i\pm1/2,j}, \kappa^2_{i,j\pm1/2}, p_{i,j}$ are determined easily of the reconstructed persymmetric matrix elements via simple linear relations.

The developed algorithm can be used for reconstructing another discrete equations with prescribed symmetry conditions. In particular, we succeeded to calculate the potential for the two-dimensional discrete Schrödinger equation. The problem is reduced to reconstructing a persymmetric five-diagonal matrix with constant (equal to -1) outdiagonal elements. This matrix has $L < MN$ different diagonal elements and can be reconstructed of L given eigenvalues.

References

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