Isospin Symmetry Breaking and Electromagnetic Effects in $K^+ \to 3\pi$ and K_{e4} Decays

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Abstract

The Fermi—Watson theorem is generalized to the case of two coupled channels with different masses. The proposed approach is applied to a final state interaction in K_{e4} decay but can be easily generalized for an arbitrary two channel task. The final state interactions in $K^{\pm} \to \pi^{\pm}\pi^0\pi^0$ decays are also considered using methods of non-relativistic quantum mechanics. The electromagnetic corrections are given as evaluated in a potential model. The obtained results are crucial for scattering lengths extraction from experimental data on K_{e4} and $K^+ \to 3\pi$ decays.

The experiments DIRAC and NA48/2 at CERN SPS are able to measure the $\pi\pi$ scattering lengths difference $a_0 - a_2$. Its value is predicted by Chiral Perturbation Theory (ChPT) with a high accuracy ($a_0 - a_2 = 0.265 \pm 0.004$ in units of inverse pion mass.) Thus, the extraction of a_0, a_2 from experimental data with comparable precision becomes an important task.

The usual method used for extraction of the scattering length a_0 from K_{e4} decays i.e.

$$K^{\pm} \to \pi^+\pi^- e^{\pm}\nu, \quad K^{\pm} \to \pi^0\pi^0 e^{\pm}\nu \tag{1}$$

is based on the classical work [1]. The decay rates of K_{e4} are determined by three form factors F, G, H. Making the partial-wave expansion of the hadronic current with respect to the angular momentum of the dipion system and restricted to s and p waves these form factors can be cast in the following form:

$$F = f_s e^{i\delta_0^0(s)} + f_p e^{i\delta_1^1(s)} \cos\theta_\pi, \quad G = g_p e^{i\delta_1^1(s)}, \quad H = h_p e^{i\delta_1^1(s)}. \tag{2}$$

Here $s = M_{\pi\pi}^2$, θ_π are invariant mass squared of the dipion and the polar angle of pion in the dipion rest frame measured with respect to the flight direction of dipion in the K meson rest frame. The coefficients f_s, f_p, g_p, h_p can be parameterized as functions of pions momenta q in a dipion rest system and invariant mass of lepton pair $s_{e\nu}$ in a known way. The phases δ_l^I relevant to certain isospin I and orbital momenta l of dipion system due to Fermi—Watson theorem [2] coincide with the corresponding phase shifts in elastic $\pi\pi$ scattering.

Recently, in experiment NA 48/2 at CERN in the $\pi^0\pi^0$ mass distribution from the decays $K^{\pm} \to \pi^{\pm}\pi^0\pi^0$ the effect of cusp was observed which, as pointed by N. Cabibbo [3], is the result of the isospin breaking in a final state $\pi^0\pi^0$ interaction, provided by inelastic $\pi\pi$ reactions and difference in masses of neutral and charge pions. Obviously, the same effects can take place in K_{e4} decays. Up to now the final state interactions of two pions in K_{e4} decay are considered using the Fermi—Watson theorem [2] which is valid only in the isospin symmetry limit i.e. at $m_c = m_0$. In the present work we study a generalization of a usually accepted approach to K_{e4} decays taking into account the inelastic processes in the final state and different masses of neutral and charged pions (part 1). We also consider the electromagnetic effects in the $K^+ \to 3\pi$ decay (part 2).

1. Final state interactions and isospin breaking. First of all, let us note that the phase shift δ_0^0 which is connected with scattering length a_0^0, has impact only on hadronic form factor F whereas the form factors G and H depend only on p-wave phase shift δ_1^1. If as usual one confines to consideration of only lowest s and p waves the inelastic process $\pi^+\pi^- \to \pi^0\pi^0$ and vice versa are forbidden for $\pi\pi$ system in $l = 1$ states due to identity of neutral pions. Thus, the inelastic transitions can change only the first term in the form factor F which is relevant to production of s-wave pions in the state with isospin $I = 0$.

Keeping this in mind, let us denote by M_{+-} the decay amplitude corresponding to two charged pions in the final state with quantum numbers $I = 0$, $l = 0$ whereas the amplitude for K meson decay to two neutral pions in the dipion state $I = 0$, $l = 0$ we denote as M_{00}. It is easy to show that in a one-loop approximation of the nonperturbative effective field theory these amplitudes take the form:

$$\begin{aligned} M_{00} &= \tilde{M}_{00}(1 + ik_1a_{00}) + ik_2a_x\tilde{M}_{+-} \\ M_{+-} &= \tilde{M}_{+-}(1 + ik_2a_{+-}) + ik_1a_x\tilde{M}_{00}. \end{aligned} \tag{3}$$

Here, $\tilde{M}_{00}$, $\tilde{M}_{+-}$ are the so-called "unperturbed" amplitudes of decays (1); $2k_1 = \sqrt{M^2 - 4m_0^2}$, $2k_2 = \sqrt{M^2 - 4m_c^2}$ are the momenta of $\pi^0\pi^0$ and $\pi^+\pi^-$ systems with the same invariant mass M. The a_{00}, a_{+-}, a_x are the s-waves amplitudes of the reactions $\pi^0\pi^0 \to \pi^0\pi^0$, $\pi^+\pi^- \to \pi^+\pi^-$ and $\pi^+\pi^- \to \pi^0\pi^0$ respectively. In the isospin symmetry limit they are connected with scattering lengths a_0, a_2 through the relations:

$$a_{00} = (a_0 + 2a_2)/3, \quad a_{+-} = (2a_0 + a_2)/3, \quad a_x = \sqrt{2}(a_0 - a_2)/3. \tag{4}$$

From the rule $\Delta I = 1/2$ for semileptonic decays it follows a simple relation between the "unperturbed" amplitudes $\tilde{M}_{+-} = \sqrt{2}\tilde{M}_{00}$. Using this fact, one obtains from the expressions (3) in the isospin symmetry limit ($k_1 = k_2$):

$$\begin{aligned} M_{00} &= \tilde{M}_{00}(1 + ika_0) = \tilde{M}_{00}\sqrt{1 + k^2a_0^2}\, e^{i\delta_0^0}, \\ M_{+-} &= \tilde{M}_{+-}(1 + ika_0) = \tilde{M}_{+-}\sqrt{1 + k^2a_0^2}\, e^{i\delta_0^0}. \end{aligned} \tag{5}$$

The considered picture can be generalized to higher orders. Summing all subsequent loops of $\pi\pi$ scattering, we obtain:

$$M_{00} = [\tilde{M}_{00}(1 - ik_2a_{+-}) + ik_2a_x\tilde{M}_{+-}]/D, \quad M_{+-} = [\tilde{M}_{+-}(1 - ik_1a_{00}) + ik_1a_x\tilde{M}_{00}]/D,$$

$$D = (1 - ik_1a_{00})(1 - ik_2a_{+-}) + k_1k_2a_x^2. \tag{6}$$

It is convenient to rewrite these equations in the form:

$$\begin{aligned} M_{00} &= \tilde{M}_{00}\sqrt{1 + k_2^2(a_{+-} - \sqrt{2}a_x)^2}\, e^{i\delta_{00}}/|D|, \\ M_{+-} &= \tilde{M}_{+-}\sqrt{1 + k_1^2(a_{00} - a_x/\sqrt{2})^2}\, e^{i\delta_{+-}}/|D|, \end{aligned} \tag{7}$$

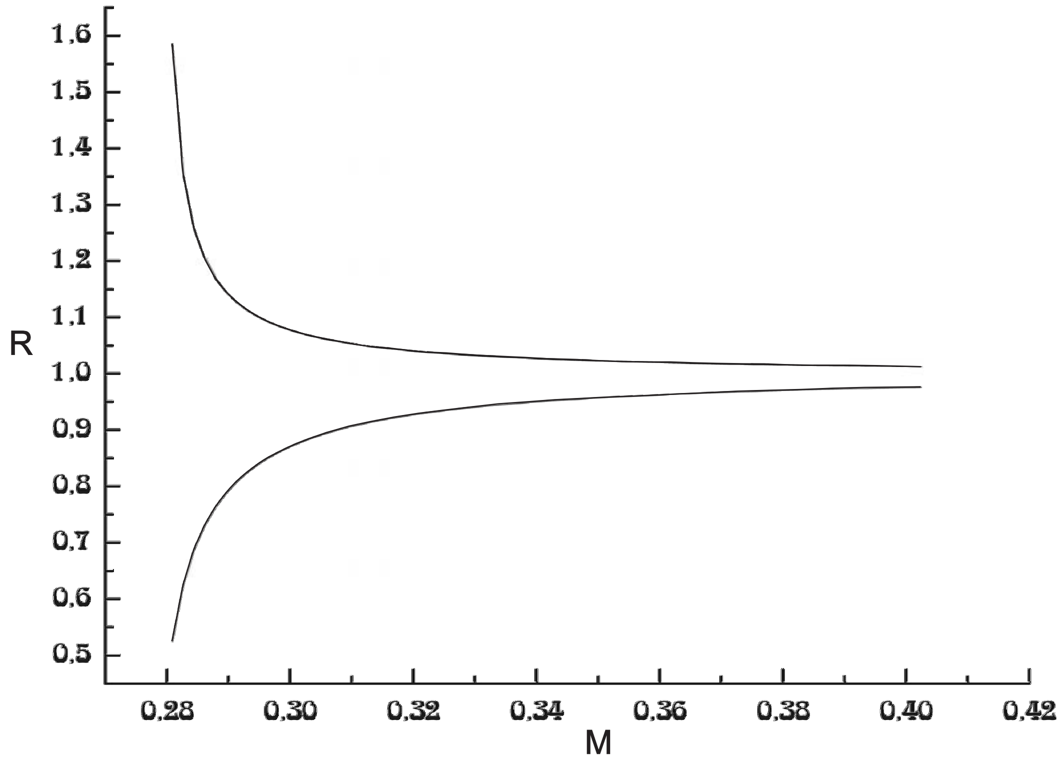

Fig. 1: The dependence of phase shifts ratios for $\pi^+\pi^-$ (upper curve) and $\pi^0\pi^0$ (lower curve) on invariant mass of pion pair

$$\begin{aligned}
\delta_{00} &= \arctan \frac{k_1 a_{00} + k_2 a_{+-}}{1 + k_1 k_2 (a_x^2 - a_{00} a_{+-})} - \arctan k_2 (a_{+-} - \sqrt{2} a_x), \\
\delta_{+-} &= \arctan \frac{k_1 a_{00} + k_2 a_{+-}}{1 + k_1 k_2 (a_x^2 - a_{00} a_{+-})} - \arctan k_1 (a_{00} - a_x/\sqrt{2}).
\end{aligned} \tag{8}$$

In the case of exact isospin symmetry ($m_c = m_0$) these equations become:

$$M_{00} = \frac{\tilde{M}_{00}}{\sqrt{1 + k^2 a_0^2}} \, e^{i\delta_0^0} \tag{9}$$

which is nothing else than Fermi—Watson theorem.

Let us note that unlike the common wisdom (9) generalized decay amplitudes (7) depend not only on a_0, but also on a_2.

The expressions (7,8) completely solve the problem of generalization of Fermi—Watson theorem to the case of two coupled channels with different masses in the final state [4].

To estimate the numerical difference of the proposed approach from the usually accepted one, let us consider the ratios

$$R_{00} = \frac{\tan \delta_{00}}{k_1 a_0}, \quad R_{+-} = \frac{\tan \delta_{+-}}{k_2 a_0}.$$

The dependence of R on dipion invariant mass is depicted on Fig. 1. As one can see from the figure, the isospin breaking effects drastically change phase shifts particularly in the vicinity of charged pions production threshold $M_{\pi\pi} = 2m_c$.

2. Electromagnetic effects in the $K^+ \to \pi^+\pi^0\pi^0$ decay. The decays $K \to 3\pi$ i.e.

$$
\begin{aligned}
K^{\pm} &\to \pi^{\pm}\pi^0\pi^0, &(10)\\
K^{\pm} &\to \pi^{\pm}\pi^+\pi^- &(11)
\end{aligned}
$$

give a valuable information on the S wave $\pi\pi$ scattering lengths a_0 and a_2 for isospin 0 and 2 respectively. Recently, in the NA 48/2 experiment at $CERN\ SPS$ the high quality data on decay (10) have been obtained. The dependence of the decay rate on invariant mass of neutral pions $M^2 = (p_1+p_2)^2$ reveals an anomaly ("cusp") at the threshold of two charged pions $M^2 = 4m^2$. N. Cabibbo proposed a simple re-scattering model [3] in which the "cusp" is an effect due to charge exchange scattering process $\pi^+\pi^- \to \pi^0\pi^0$ in decay (11). The amplitude of the decay (10) consists of two terms [3]:

$$T = T_0 + 2ika_xT_+, \tag{12}$$

where T_0, T_+ are "unperturbated" amplitudes for decays (10) and (11), respectively, and $k = \sqrt{M^2-4m^2}/2$ is the momenta of the charged pion. The second term in (12) is proportional to the difference of scattering lengths $a_x = (a_0 - a_2)/3$ and flips from dispersive to absorptive at the charged pions threshold. As a result, the decay probability under threshold depends on the scattering lengths difference linearly which allows one to extract the S wave scattering lengths from experimental data with high accuracy.

Using the methods of nonrelativistic quantum mechanics, it can be shown that the result of N. Cabibbo [3] can be generalized accounting the $\pi\pi$ scattering in the "charge-exchange" part of the amplitude (second term in (12)) to all orders in scattering lengths

$$
\begin{aligned}
T &= T_0 + 2ikf_xT_+, \quad f_x = a_x/D,\\
D &= (1-ik_1R_{11})(1-ik_2R_{22}) + k_1k_2R_{12}^2. &(13)
\end{aligned}
$$

Here, $k_1 = \sqrt{M^2-4m_0^2}/2$, $k_2 = k = \sqrt{M^2-4m^2}/2$ are the neutral and charged pions momenta respectively. The elements of the R matrix are real and can be expressed through the common combinations of scattering lengths [5] $a_x = (a_2 - a_0)/3$, $a_{00} = (a_0 + 2a_2)/3$, $a_{\pm} = (2a_0 + a_2)/6$ corresponding to inelastic and elastic pion-pion scattering

$$R_{12} = \sqrt{2}a_x, \quad R_{11} = a_{00}, \quad R_{22} = 2a_{\pm}. \tag{14}$$

The replacement $a_x \to f_x$ has a small numerical impact on results of the previous calculations [5] in the dominant part of the phase space but as we will see later is very crucial for the inclusion of the electromagnetic interactions under threshold were the formation of bound states ($\pi^+\pi^-$ atoms) takes place.

A next step of our prescription is the inclusion in the expression (13) properly electromagnetic effects. The general receipt is known for many years and implies the replacement of the charged pion momenta k by a logarithmic derivative of the pion wave function in the Coulomb potential at the boundary of strong field r_0 i.e.

$$ik \to \tau = \frac{d\log[G_0(kr) + iF_0(kr)]}{dr}\bigg|_{r=r_0}. \tag{15}$$

Here, F_0, G_0 are the regular and irregular solutions of the Coulomb problem.

In the region $kr_0 \ll 1$, where the electromagnetic effects can be significant, the above replacement gives

$$\begin{aligned}
\tau &= ik - \alpha m \left[\log(-2ikr_0) + 2C + \psi\left(1 - \frac{im\alpha}{2k}\right)\right] = Re(\tau) + iIm(\tau), \\
Re(\tau) &= -\alpha m \left[\log(2kr_0) + 2C + Re\,\psi\left(1 - \frac{im\alpha}{2k}\right)\right], \\
Im(\tau) &= ikA^2, \quad A = \exp\left(\frac{\pi\xi}{2}\right)|\Gamma(1+i\xi)|, \quad \xi = \frac{\alpha m}{2k},
\end{aligned} \tag{16}$$

where $C = 0.577$, $\alpha = 1/137$ are Euler and fine structure constants, whereas ψ is the digamma function. To go under threshold, it is enough to do a common replacement $k \to i\kappa$ in the above expression

$$\tau = -\kappa - \alpha m \left[\log(2\kappa r_0) + 2C + \psi\left(1 - \frac{m\alpha}{2\kappa}\right)\right]. \tag{17}$$

At $\kappa_n = \alpha m/(2n)$, where n is an integer, τ goes to infinity which corresponds to Coulomb bound states in the considered approach. On the other hand, the product κf_x defining the amplitude behaviour under threshold remains finite due to the presence of dependence from τ in the denominator D in (13). This explains why the inclusion of electromagnetic effects can be done only after sum up all terms of infinite series in the perturbation expansion.

It is easy to see that the product κf_x possesses a resonance structure placed at the positions

$$\begin{aligned}
M_n &= 2m - \frac{\bar{\kappa}_n^2}{m}, \quad \bar{\kappa}_n = \frac{\alpha m}{2(n-\delta)}, \\
\delta &= \frac{1}{\pi}\arctan\Delta, \quad \Delta = \alpha m\left[a_{22} - \frac{k_1^2 a_{11} a_{12}^2}{1 + k_1^2 a_{11}^2}\right], \\
\Gamma_n &= \frac{4\pi k_1 a_{12}^2 \bar{\kappa}_n^3}{m(1 + k_1^2 a_{11}^2)}.
\end{aligned} \tag{18}$$

The physical reason of resonance origin is transparent. Due to the process $\pi^+\pi^- \to \pi\pi$ the Coulomb bound states of the $\pi^+\pi^-$ system ($A_{2\pi}$ atoms) becomes unstable. The considered effect of $A_{2\pi}$ atoms creation in decay (10) is not the only one contribution from the electromagnetic interaction of pions. Outside of the resonance region the Coulomb interaction leads to the essential difference between the τ values calculated with electromagnetic corrections and without it. In particular, the nonzero contribution of the Coulomb corrections to the $Re\,\tau$ above the threshold leads to the interference term in decay rate provided by "direct" and "charge-exchange" contributions from (13). Thus, above the threshold the interference is nonzero even at the lowest order in scattering lengths unlike the original approach proposed by N. Cabibbo [3].

The further improvement of the theory consists in account of the final state interaction in the "direct" term from (13). This can be done by simple substitution

$$T_0 \to T_0(1 + ik_1 f_{00}), \tag{19}$$

where f_{00} is the amplitude of $\pi^0\pi^0$ scattering. It can be shown that

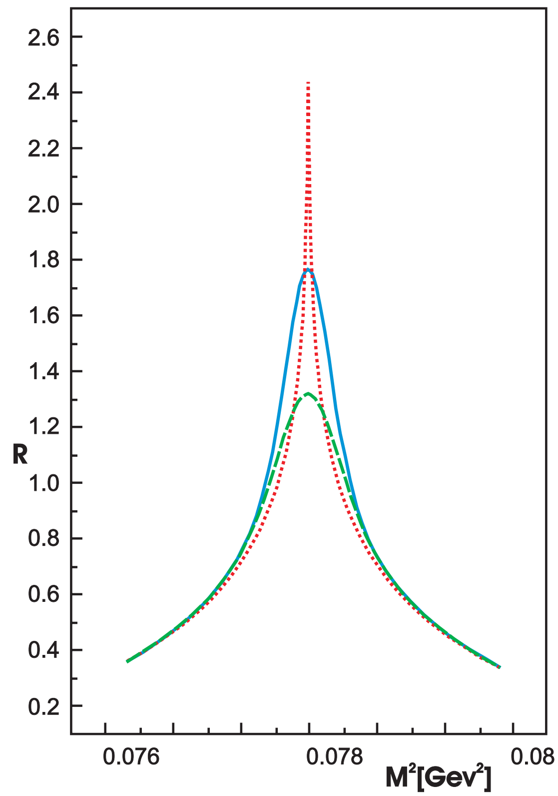

Fig. 2: The dependence of R on the square of invariant mass of charged pions

$$1 + ik_1 f_{00} = \frac{1 + \tau a_{22}}{(1 - ik_1 R_{11})(1 - \tau a_{22}) - ik_1 \tau a_{12}^2} \,. \tag{20}$$

To estimate the contribution from electromagnetic effects to decay rate of process (10) let us introduce the ratio $R(\%) = (|T_c|^2 - |T|^2)/|T|^2$, where the amplitude T is given by (13) while the amplitude T_c accounting for electromagnetic effects is given by expressions (13,19) with the relevant modifications discussed above. The dotted line on the Fig. 2 gives the contribution of electromagnetic effects without bound states (pionium) corrections. The dashed line represents the same quantity but with a corresponding amplitude averaged using gaussian distribution with expected mass resolution $r.m.s. =$ 0.56 Mev. The solid line gives the contribution of all electromagnetic effects (bound states included) averaged as in the previous case [6].

From this plot one concludes that after relevant averaging the main contribution to the decay rate comes from the electromagnetic interactions that do not lead to bound states.

References

[1] N. Cabibbo and A. Maksymowicz, Phys. Rev. B **438**, 137 (1965).

[2] K. M. Watson, Phys. Rev. **88**, 1163 (1952).

[3] N. Cabibbo, Phys. Rev. Lett. **93**, 121801 (2004).

[4] S. R. Gevorkyan, A. N. Sissakian, A. V. Tarasov, H. T. Torosyan and O. O. Voskresenskaya, hep-ph/0704.2675.

[5] N. Cabibbo and G. Isidori, JHEP **03**, 021 (2005).

[6] S. R. Gevorkyan, A. V. Tarasov and O. O. Voskresenskaya. Phys. Lett. B **649**, 159 (2007).