Numerical Investigation of Anisotropically Driven Fully Developed Turbulence

E.A. Hayryan, E. Jurcisinova
Laboratory of Information Technologies, JINR
M. Jurcisin
Laboratory of Theoretical Physics, JINR
M. Stehlik
Institute of Experimental Physics, Slovak Academy of Sciences, Košice

Abstract

The fully developed turbulence with axial anisotropy for dimensions $d > 2$ is investigated by means of a renormalization group approach. The corresponding system of strongly nonlinear renormalization group equations is solved numerically. A possible utilization of the parallel programming methods is discussed. As a result, the influence of anisotropy on the stability of the Kolmogorov scaling regime is analyzed. The borderline dimension between the stable scaling regime and unstable one is calculated as a function of the anisotropy parameters.

One suitable tool for investigation of fully developed turbulence based on the stochastic Navier-Stokes equation is the so-called quantum-field renormalization group (RG) [1]. In early papers, the RG approach was applied only to isotropic models of developed turbulence. However, the method can also be used in the theory of anisotropically developed turbulence. The reason for theoretical study of the influence of anisotropy on the behavior of developed turbulence is given by the fact that a variety of experimental studies as well as computer simulations indicate the existence of deviation from the isotropic statistics of the fully developed turbulence. A question immediately arises here: whether the scaling regime remains stable under transition from the isotropic to the anisotropic case. In other words, do the stable fixed points of the RG equations remain stable under the influence of anisotropy?

In the statistical theory of anisotropically developed turbulence, the turbulent flow is characterized by the random velocity field $\mathbf{v}(\mathbf{x}, t)$, where $\mathbf{v}$ and $\mathbf{x}$ are supposed to be d-dimensional vectors. Its evolution is governed by the randomly forced Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu_0 \Delta \mathbf{v} - \mathbf{f}^A = \mathbf{f}, \qquad (1)$$

where the incompressibility of the fluid is assumed, which is given mathematically by the conditions $\nabla\cdot\mathbf{v} = 0$ and $\nabla\cdot\mathbf{f} = 0$. The parameter ν_0 is the kinematic viscosity (subscript 0 denotes bare parameters, see, e.g., Ref. [1]). The term $\mathbf{f}^{\mathbf{A}}$ is related to uniaxial anisotropy, and it has the following form [2]

$$\mathbf{f}^{\mathbf{A}} = \nu_0 \left[\chi_{10}(\mathbf{n}\nabla)^2\mathbf{v} + \chi_{20}\mathbf{n}\nabla^2(\mathbf{n}\mathbf{v}) + \chi_{30}\mathbf{n}(\mathbf{n}\nabla)^2(\mathbf{n}\mathbf{v})\right]. \qquad (2)$$

The parameters χ_{10}, χ_{20} and χ_{30} characterize the weight of the individual structures in Eq. (2), and the unit vector $\mathbf{n}$ specifies the direction of the anisotropy axis. The large-scale random force per unit mass $\mathbf{f}$ is assumed to have Gaussian statistics defined by the averages

$$\langle f_i \rangle = 0,\ \langle f_i(\mathbf{x}_1, t) f_j(\mathbf{x}_2, t)\rangle = D_{ij}(\mathbf{x}_1 - \mathbf{x}_2, t_1 - t_2). \qquad (3)$$

The two point correlation matrix

$$D_{ij}(\mathbf{x},t) = \delta(t)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\tilde{D}_{ij}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x}) \tag{4}$$

is convenient to parametrize in the following way [2]

$$\tilde{D}_{ij}(\mathbf{k}) = g_0\nu_0^3 k^{4-d-2\epsilon}[(1+\alpha_1\xi_k^2)P_{ij}(\mathbf{k}) + \alpha_2 R_{ij}(\mathbf{k})]\,, \tag{5}$$

where a vector $\mathbf{k}$ is the wave vector, d is the dimension of the space (in our case: $2 < d$), $\epsilon \geq 0$ is dimensionless parameter of the model. The physical value of this parameter is $\epsilon = 2$ (the so-called energy pumping regime). The value $\epsilon = 0$ corresponds to a logarithmic perturbation theory for a calculation of Green functions when g_0, which plays the role of a bare coupling constant of the model, becomes dimensionless [1]. The $(d \times d)$-matrices P_{ij} and R_{ij} are the transverse projection operators. Their explicit form is

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}\,,\quad R_{ij}(\mathbf{k}) = \left(n_i - \xi_k\frac{k_i}{k}\right)\left(n_j - \xi_k\frac{k_j}{k}\right)\,, \tag{6}$$

where ξ_k is given by the equation $\xi_k = \mathbf{k}\cdot\mathbf{n}/k$. The tensor $\tilde{D}_{ij}$, given by Eq.(5), is the most general form with respect to the condition of incompressibility of the system under consideration and contains two dimensionless free parameters α_1 and α_2. The positiveness of the correlator tensor D_{ij} leads to restrictions on the above parameters, namely, $\alpha_1 > -1$ and $\alpha_2 > -1$.

The stochastic problem (1) with correlator (4) can be transformed into the field theoretic model of fields $\mathbf{v}$ and $\mathbf{v}'$, where $\mathbf{v}'$ is independent of the velocity field $\mathbf{v}$ auxiliary incompressible field, which we have to introduce when transforming the stochastic problem into a functional form. After this transformation the action of the fields $\mathbf{v}$ and $\mathbf{v}'$ is given in the form

$$\begin{aligned} S &= \frac{1}{2}\int d^d\mathbf{x}_1 dt_1 d^d\mathbf{x}_2 dt_2 \left[v_i'(\mathbf{x}_1,t_1)D_{ij}(\mathbf{x}_1-\mathbf{x}_2,t_1-t_2)v_j'(\mathbf{x}_2,t_2)\right] \\ &+ \int d^d\mathbf{x}dt\left\{\mathbf{v}'(\mathbf{x},t)\left[-\partial_t\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v} + \nu_0\nabla^2\mathbf{v} + \mathbf{f}^A\right](\mathbf{x},t)\right\}\,. \end{aligned} \tag{7}$$

The functional formulation gives the possibility to use the quantum field theory methods, including the RG technique, to solve the problem. The formulation through action functional (7) replaces the statistical averages of random quantities in the stochastic problem (1)-(5) with equivalent functional averages with weight $\exp[S(\mathbf{v},\mathbf{v}')]$ (for details see [1]).

The fully developed turbulence is characterized by the large Reynolds number Re. On the other hand, the large Re corresponds to the existence of a large inertial interval, which is defined by the inequalities $1/\Lambda = l \ll r \ll L = 1/m$, where l corresponds to an inner scale (the scale where dissipation forces are dominated, or the scale of the smallest eddies), and L is an outer scale of the system (the scale of the energy pumping into the system, or the scale of the largest eddies). In the fully developed turbulence we are interested in the behavior of the correlation functions of velocity field $\langle v_{i_1}(\mathbf{x}_1,t),...,v_{i_N}(\mathbf{x}_N,t)\rangle$ deep inside of the inertial interval, i.e., far away from the dissipation effects as well as far away from energy pumping scale. Within the field theoretic approach they are given by the following functional integral

$$\langle v_{i_1}(\mathbf{x}_1,t),...,v_{i_N}(\mathbf{x}_N,t)\rangle = \int \mathcal{D}\Phi\; v_{i_1}(\mathbf{x}_1,t),...,v_{i_N}(\mathbf{x}_N,t)e^{S(\Phi)}\,, \tag{8}$$

where $\Phi = \{\mathbf{v}, \mathbf{v}'\}$, $1 \leq i_j \leq d, j = 1, ..., N$, and $S(\Phi)$ is given by Eq. (7).

The behavior of the correlation functions inside the inertial interval is the main issue of the famous Kolmogorov-Obukhov phenomenological theory (see also Ref. [3]). It was formulated in the form of two hypotheses which lead to the scaling behavior of the correlation functions within the inertial interval. In what follows we shall discuss only the so-called second Kolmogorov hypothesis related to the IR scaling and our aim will be to investigate the influence of the axial anisotropy on this scaling behavior.

Within the RG technique the correlation functions are obtained directly in the scaling form (with correct critical dimensions) and their large-scale limit (i.e., IR limit) is described by the stable fixed points of the renormalization theory, i.e., the scaling regime is stable if the corresponding fixed point is IR stable. The IR fixed point is obtained by using the system of differential equations (also called the flow equations) which drive the effective variables $\bar{C} = \{\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3\}$ which are the functions of dimensionless scale parameter (wave number) $t = k/\Lambda$. Their explicit form is the following

$$t\frac{d\bar{g}}{dt} = \beta_g(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d)\,, \tag{9}$$

$$t\frac{d\bar{\chi}_i}{dt} = \beta_{\chi_i}(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d)\,, \quad i = 1, 2, 3\,. \tag{10}$$

The dimensionless wave number t belongs to the interval $0 \leq t \leq 1$, and the initial conditions for the above differential equations are taken at $t = 1$. The IR stable fixed point corresponds to the values in the limit $t \to 0$, i.e., $(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3)|_{t\to 0} = (g^*, \chi_1^*, \chi_2^*, \chi_3^*)$ (standardly, a quantity with star denotes the fixed point value). The so-called β-functions $\beta_g, \beta_{\chi_i}, i = 1, 2, 3$ are defined by the so-called renormalization constants of the renormalization procedure and their final form is as follows (details see in Ref. [4])

$$\beta_g = g(-2\epsilon + 3Aga_1)\,, \quad \beta_{\chi_i} = -Ag\,(a_{i+1} - \chi_i a_1)\,, \quad i = 1, 2, 3\,, \tag{11}$$

where parameter A is defined as $A = S_{d-1}/((2\pi)^d(d^2 - 1))$, S_d is the area of the d-dimensional sphere given as $S_d = 2\pi^{(d/2)}/\Gamma(d/2)$, and functions a_i $(i = 1, ..., 4)$ are given in Ref. [5].

The most comfortable way how for obtaining of the IR fixed point of the system of four differential RG equations (9) and (10) with (11) is to solve it numerically using some appropriate numerical method. In what follows, we work with the fourth-order Runge-Kutta method with the adaptive choice of the integration step. For this purpose it is convenient to transform the system of differential equations (9) and (10) into an autonomic system by substitution $t = e^{-s}$. Using this transformation one obtains

$$\frac{d\bar{g}}{ds} = -\beta_g(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d)\,, \tag{12}$$

$$\frac{d\bar{\chi}_i}{ds} = -\beta_{\chi_i}(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d)\,, \quad i = 1, 2, 3\,, \tag{13}$$

where $s \in [0, \infty)$. The initial conditions correspond to $s = 0$ and the IR fixed point is found in the limit $s \to \infty$. The first step for the variable s was taken as $\Delta s = 10^{-3}$. The initial values of the parameters can be chosen arbitrary but the most convenient choice is to take them to be the fixed point of the three-demensional isotropic model (see Ref. [2]).

Now we have all necessary tools at hand to investigate the fixed point and its stability. Our aim is to find the so-called borderline dimension d_c between stable and unstable regimes when moving from the three-dimensional system towards two-dimensional one. In Figs. 1 and 2 our results for d_c as a function of the anisotropy parameters are present. Figs. 1 and 2 show that in the three-dimensional system the Kolmogorov scaling regime is unstable only in the limit $\alpha_{1,2} \to -1$ and for large enough values of parameter α_1 together with negative or relatively small positive values of the parameter α_2. Our conclusion is the following: to destroy stability of the Kolmogorov scaling regime in three-dimensional space by the uniaxial anisotropy, which is in our model represented by the parameters α_1, and α_2, it is necessary to apply anisotropy with rather specific values of these parameters.

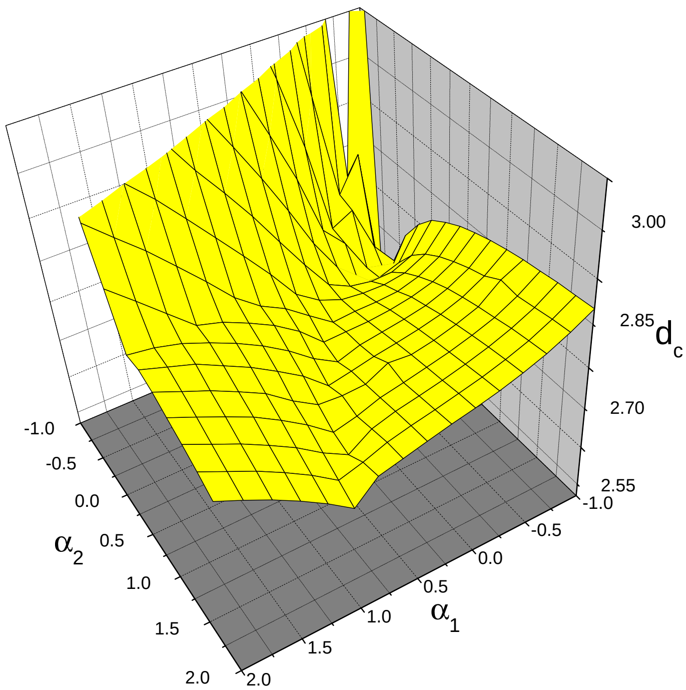

Fig. 1: The three-dimensional view on the dependence of the borderline dimension d_c on the parameters α_1, and α_2

An important question is related to the choice of a numerical method for calculation of the integrals. It can be shown that the most appropriate method is the using of the Chebyshev quadrature formula. The question of the number of divisions of the integration interval is another important one. In our calculations, we used the division to 1024 subintervals, which was found as the best choice from the point of view of the accuracy and needed time of the calculation. On the other hand, in Ref. [2], the division to the 128 subintervals was used. We suppose that this fact could lead to the difference between our and their results because the division to 128 subintervals may not be sufficient in some critical situations.

To find the borderline dimension it is enough to use the bisection method. Our results was calculated with the accuracy of 0.005. The same accuracy was supposed in Ref. [2] but, as can be seen from their results, this accuracy was not achieved by them even in the isotropic limit where exact result exists.

From numeric calculations point of view, the problem is rather time-consuming, i.e., the calculations take relatively long time. Therefore, the question of using a modern computational methods arises. We have analyzed possible speed up of calculations based on the utilization of the parallel programming methods using the Message Passing Interfa-

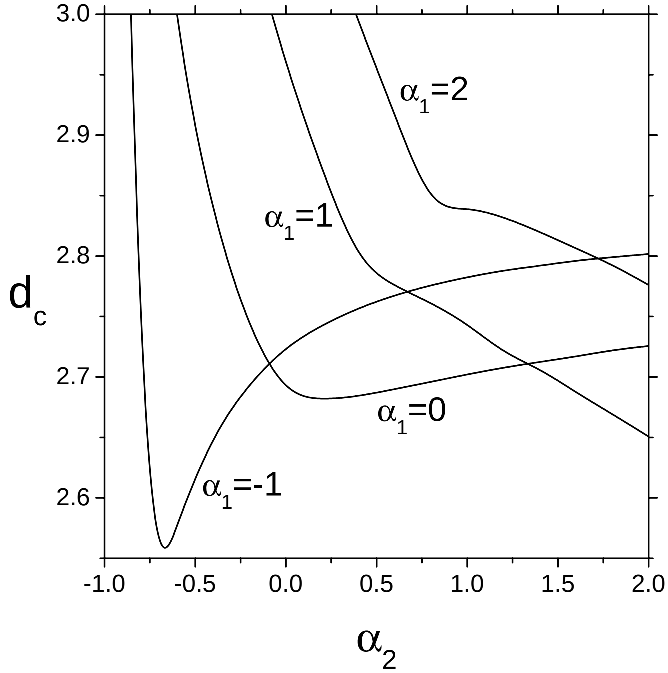

Fig. 2: The dependence of the borderline dimension d_c on the parameter α_2 for concrete values of the parameter α_1

ce (MPI) (details see in Ref. [5]). In concrete calculations we used advantage of the parallel programming.

Conclusions: By using the field-theoretic RG method the influence of the uniaxial anisotropy on the stability of the Kolmogorov scaling regime in fully developed turbulence was investigated. The stability of the regime is defined by the very existence of the IR stable fixed point. The fixed point was found numerically by the solving of the corresponding differential RG equations. We have found that the stability of the three-dimensional scaling regime is destroyed only in the case of rather large (in the sense of the absolute value) and special values of the anisotropy parameters. We have also analyzed the optimal way how to calculate the numerical problem by using the parallel programming methods.

References

[1] L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasiliev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon & Breach, London, 1999).

[2] J. Busa, M. Hnatich, J. Honkonen and D. Horvath, Phys. Rev. E **55**, 381 (1997).

[3] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. 2.

[4] E. A. Hayryan, E. Jurcisinova, M. Jurcisin, I. Pokorny and M. Stehlik, Communication of JINR E17-2005-208 (2005).

[5] E. A. Hayryan, E. Jurcisinova, M. Jurcisin and M. Stehlik, Mathematical Modelling and Analysis 12, 325 (2007).