Hard Transition from a Stationary State to Oscillations for a Linear Differential Equation

S.I. Serdyukova
Laboratory of Information Technologies, JINR

In [1] we show the existence of slowly damping oscillations of the solution of the linear hyperbolic equations

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}, \tag{1}$$

with the discontinuous initial data

$$u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0, \end{cases} \qquad u_t(x,0) = 0. \tag{2}$$

These oscillations are excited for $|x| > t$. In rhe strip between the characteristics $|x| < (1-\delta)t$ the solution is exponentially close to $1/2$. This problem arises in the study of wave propagation in periodic stratified media [2],[3].

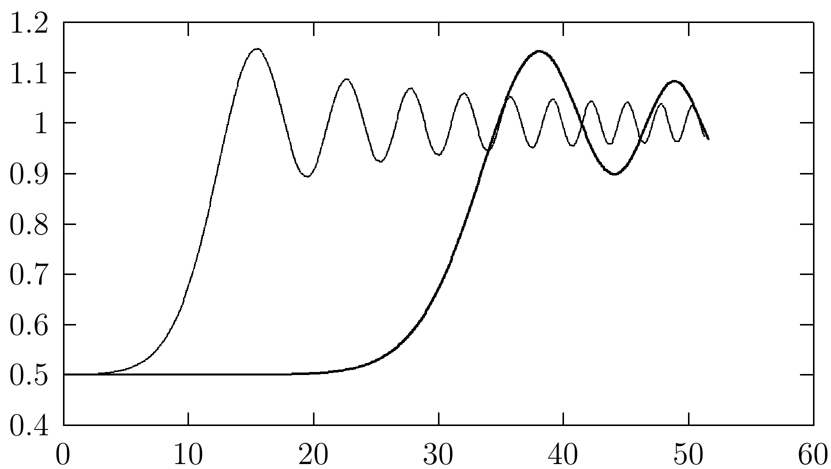

Fig. 1.

Following to M.V. Fedoryuk [4], we construct an integral representation of the solution of the problem under study:

$$u = -\frac{1}{4\pi i} \int\limits_{\Gamma} \left(\exp(-ist\sqrt{1+s^2} - ixs) + \exp(ist\sqrt{1+s^2} - ixs) \right) \frac{ds}{s}.$$

The contour Γ goes along the real line, except a neighborhood of zero: the pole is rounded in the upper half plane over an arc σ that is a semicircle of small radius.

In [1] we derive the asymptotics of the solution to problem (1), (2) as $t \to \infty$ for all x. We prove seven theorems describing the behavior of the solution for various values of x. The theorems are proved using methods of the theory of functions of a complex variable, in particular, the saddle point method [5]. We formulate the principle theorem of [1].

Theorem 2. *As* $t \to \infty$, *for* $x > t^{1+\delta}$,

$$u(x,t) = 1 + \frac{\sin((x^2/4t)(1+O(t^{-2\delta})) - \pi/4)}{\sqrt{\pi x^2/t}}(1 + O(t^{-1} + (x/t)^{-4})) + O(\exp(-x)).$$

The following two theorems of [1] prove that as $t \to \infty$, $0 < x < t^{1-\delta}$, the solution of the problem differs from 1/2 by an exponentially small value in t. These theoretical results are in good agreement with numerical experiments.

Fig. 1 shows the numerical solutions for $0 < x < 55$ at $t = 10$ and $t = 30$. The graph corresponding to $t = 10$ increases faster in the neighborhood of $x = 0$. In this graph slowly damping oscillations to the right of the characteristic $x - t = 0$, for $x > 10$, is clearly seen. In the graph corresponding to $t = 30$ there is a distinct horizontal segment to the left of the characteristic, for $x < 30$. In Fig. 2 slowly damping oscillations of $u(x, 30)$ to the right of the characteristic, for $x > 30$, can be clearly seen.

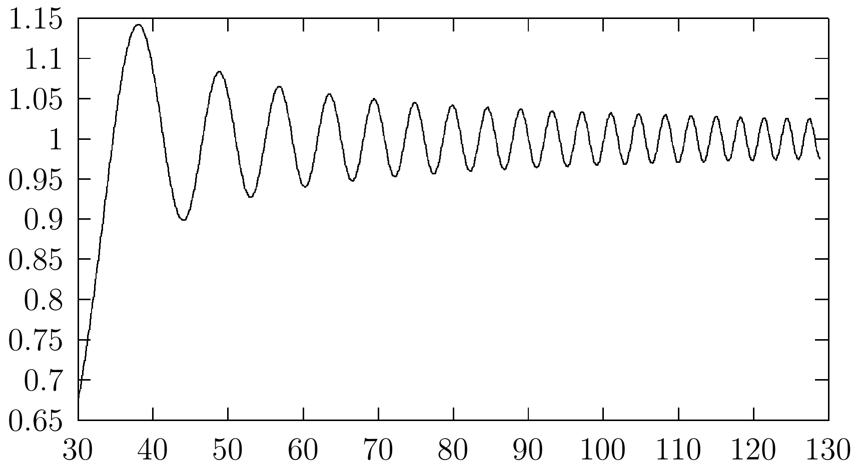

Fig. 2.

Equation (1) was approximated by the second-order accurate explicit difference scheme

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} - \frac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n}{h^4},$$

which is stable in L_2 for $\tau < h^2/\sqrt{4+h^2}$. In the computations we used $h = 0.1$ and $\tau = h^2/\sqrt{2(4+2h^2)}$.

References

[1] S.I.Serdyukova//Dokl.Math.,2007, v.76, pp.554-558.

[2] N.S.Bakhvalov, M.E.Eglit//Dokl.Math.,2000, v.61, pp.1-4.

[3] N.S.Bakhvalov, M.E.Eglit//Dokl.Math.,2002, v.65, pp.301-306.

[4] M.V.Fedoryuk//Mat. Sb.,1963,62(104), pp.397-468.

[5] N.G. de Bruijn*Asymptotic Methods in Analysis* (North-Holland.Amsterdam.1958, Inostrannaya Literatura.Moscow. 1961).