Spheroidal and Torsional Modes of Quasistatic Shear Oscillations in the Solid Globe Models of Nuclear Physics and Pulsar Astrophysics

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Abstract

Among the most important development of nuclear physics and pulsar astrophysics over the past three decades have seen gradual recognition that Fermi-matter constituting interior of atomic nuclei and main body of neutron stars possesses solid-mechanical material properties of elasticity and viscosity and can be identified, therefore, as viscoelastic Fermi-solid. This paper focuses on one issue of elastodynamical model of the continuum mechanics of nuclear matter regarding oscillatory behavior of a spherical mass of viscoelastic solid in the regime of quasistatic, force-free, non-compressional oscillations less investigated in the literature compared to oscillations in the regime of standing shear waves. We present rigorous mathematical treatment of this problem by Rayleigh's variational method and demonstrate practical usefulness of developed theory by solutions of several problems of current nuclear physics and pulsar astrophysics.

1 Introduction. The nearly linear dependence of the nucleus volume and the nucleus binding energy upon number of nucleons, meaning the saturation of internucleon forces, unambiguously indicate that this according to Bethe "most conspicuous feature of nuclei" is the same as it is for condensed matter. The condensed matter, as is known, comes in two forms – liquid and solid. Accordingly, the question which of two fundamental models of material continua – the fluid-mechanical or the solid-mechanical can provide adequate description of the observable behavior of the nuclear matter objects has been and still is central to the program on the study of material properties of superdense substance constituting interior of atomic nuclei and neutron stars.

The past three decades of investigation on nuclear physics and pulsar astrophysics have seen gradual recognition that elastodynamic approach to the continuum mechanics of nuclear matter provides proper account of macroscopic motions of degenerate Fermi-matter constituting interior of the nuclear material objects, the densest of all known today. This paper focuses on one subtle issue of general solid-mechanical treatment of oscillatory behavior of the nuclear matter objects in terms of vibrations of a viscoelastic solid globe that has been called into question by the above development of both the nuclear physics and pulsar astrophysics. Specifically, it is concerned with the regime of quasistatic, force-free, oscillations less investigated in the literature as compared to the solid globe oscillations in the regime of standing shear waves.

2 Nodeless vibrations of viscoelastic solid globe. The basic dynamical variable characterizing the state of motion of solid continuous medium is the field of material displacement $u_i(\mathbf{r},t)$ whose emergence in the volume of solid object is associated with its response to perturbation of mechanically equilibrium state in which $u_i = 0$. The second law of Newtonian dynamics for an isotropic incompressible viscoelastic medium is expressed by the Lamé-Navier equation

$$\rho\ddot{u}_i = \nabla_k\, p_{ik} + \nabla_k\, \pi_{ik}. \tag{1}$$

where p_{ik} is tensor of shear elastic and π_{ik} shear viscose stresses obeying Hookean law of elasticity and Newtonian law of viscosity

$$p_{ik} = 2\mu u_{ik} \qquad \pi_{ik} = 2\eta\dot{u}_{ik} \tag{2}$$

$$u_{ik} = \frac{1}{2}(\nabla_i u_k + \nabla_k u_i) \qquad u_{kk} = \nabla_k u_k. \tag{3}$$

From now on u_{ik} stands for tensor of deformation; μ is the shear modulus and η shear viscosity which are regarded as constants. The energy balance in the process of deformations is controlled by equation the integral equation of energy balance of viscoelastic shear deformations

$$\frac{\partial}{\partial t}\int \frac{\rho\dot{u}^2}{2}\, d\mathcal{V} \;=\; -\int p_{ik}\dot{u}_{ik}\, d\mathcal{V} - \int \pi_{ik}\, \dot{u}_{ik}\, d\mathcal{V} \tag{4}$$

$$\;=\; -2\int \mu\, u_{ik}\dot{u}_{ik} d\mathcal{V} - 2\int \eta\, \dot{u}_{ik}\, \dot{u}_{ik}\, d\mathcal{V} \tag{5}$$

In what follows the focus is laid on the oscillatory response in which field of material displacements obey the vector Laplace equation

$$\nabla^2\mathbf{u}(\mathbf{r},t) = 0 \qquad \nabla\cdot\mathbf{u}(\mathbf{r},t) = 0. \tag{6}$$

This equation can be thought of as long-wavelength limit of the vector Helmholtz equation $\nabla^2\mathbf{u} + k^2\nabla^2 = 0$ lying at the base of the well-studied standing-wave regime of oscillatory response of viscoelastic sphere: in the limit of extremely long wavelengths $\lambda \to \infty$ the wave vector $k = (2\pi/\lambda) \to 0$. The most striking feature of the regime of long wavelength oscillations is that the restoring force of Hookean elastic (reversal) stress and dissipative force of Newtonian viscous (irreversal) stress entering the basic equation of solid mechanics turn to zero (from what the term quasistatic oscillations is derived), but the material stresses themselves and the work done by these stresses in the bulk of an oscillating solid globe do not. Based on this observation we show that in the case of force-free fluctuations of stresses the frequency and lifetime of both spheroidal and torsional modes in a viscoelastic solid globe can be computed by taking advantage of the energy variational principle relying on the equation of energy conservation.

The point of departure is to use separable representation of the vector field of displacements and tensor field of shear deformations

$$u_i(\mathbf{r},t) = a_i(\mathbf{r})\,\alpha(t) \quad u_{ik}(\mathbf{r},t) = a_{ik}\,(\mathbf{r})\,\alpha(t) \quad a_{ik} = \frac{1}{2}\left(\nabla_i\, a_k + \nabla_k a_i\right), \tag{7}$$

where $\alpha(t)$ is the temporal amplitude and $\mathbf{a}(\mathbf{r})$ is the solenoidal vector field of instantaneous, time-independent, displacements. On substituting (7) in (5) we arrive at the well-familiar equations for the amplitude $\alpha(t)$:

$$\frac{\partial \mathcal{H}}{\partial t} = -2\mathcal{F} \qquad \mathcal{H} = \frac{M\dot{\alpha}^2}{2} + \frac{K\alpha^2}{2} \qquad \mathcal{F} = \frac{D\dot{\alpha}^2}{2} \tag{8}$$

$$M\ddot{\alpha} + D\dot{\alpha} + K\alpha = 0 \tag{9}$$

describing damped harmonic oscillator. The Hamiltonian $\mathcal{H}$ stands for the total energy of dissipative free, normal, vibrations and $\mathcal{F}$ is the Rayleigh's dissipative function. The integral coefficients of inertia M, stiffness K and viscous friction D are given by

$$M = \int \rho(r)\, a_i\, a_i\, d\mathcal{V} \quad K = 2\int \mu(r)\, a_{ik}\, a_{ik}\, d\mathcal{V} \quad D = 2\int \eta(r)\, a_{ik}\, a_{ik}\, d\mathcal{V}. \tag{10}$$

The solution taken in the form $\alpha(t) = \alpha_0 \exp(\lambda t)$ leads to

$$\Omega^2 = \omega^2 \left[1 - (\omega\tau)^{-2}\right] \quad \omega^2 = \frac{K}{M} \quad \tau = \frac{2M}{D}. \tag{11}$$

where ω is the frequency of the free, non-damped, oscillations and the τ is the time of their viscous damping. Thus, to compute the frequency and lifetime of quasistatic oscillations one need to specify the fields of instantaneous material displacements $\mathbf{a}(\mathbf{r})$ entering the integral coefficients M, K and D of oscillating solid globe. The radial profiles of density $\rho(r)$, the shear modulus $\mu(r)$ and shear viscosity $\eta(r)$ in the solid globe are regarded as input data of the method.

Adhering to the above Lamb's classification of the vibrational eigenmodes of a perfectly elastic solid sphere, as spheroidal (shake or $s-mode$) and torsional (twist or $t-mode$), in the case under consideration, the eigenmodes of quasistatic regime of oscillations can be specified by two fundamental solutions to the vector Laplace equation, built on the general, regular in origin, solution of the scalar Laplace equation

$$\nabla^2 \chi(\mathbf{r}) = 0 \qquad \chi(\mathbf{r}) = r^\ell P_\ell(\cos\theta). \tag{12}$$

The first one, describing instantaneous displacements in spheroidal mode of quasistatic oscillations is given by the even parity (polar) poloidal vector field

$$\mathbf{a}_s = A_s \nabla r^\ell P_\ell(\cos\theta) \tag{13}$$

exhibiting irrotational character of quasistatic spheroidal oscillations: $\nabla \times \mathbf{a}_s = 0$. The second fundamental solution to the vector Laplace equation describing instantaneous displacements in the torsional mode is given by the odd parity (axial) toroidal vector field

$$\mathbf{a}_t = A_t \nabla \times [\mathbf{r}\, \chi(\mathbf{r})] = A_t [\nabla \chi(\mathbf{r}) \times \mathbf{r}]. \tag{14}$$

The radial profiles of these fields have no nodes in volume of oscillating globe, that is, in the interval $[0 < r < R]$. In view of this, the term non-radial or nodeless shear oscillations to this type of oscillatory behavior is applied. The intrinsic distortions in a viscoelastic sphere undergoing non-radial spheroidal and torsional quasistatic oscillations are shown Fig.1.

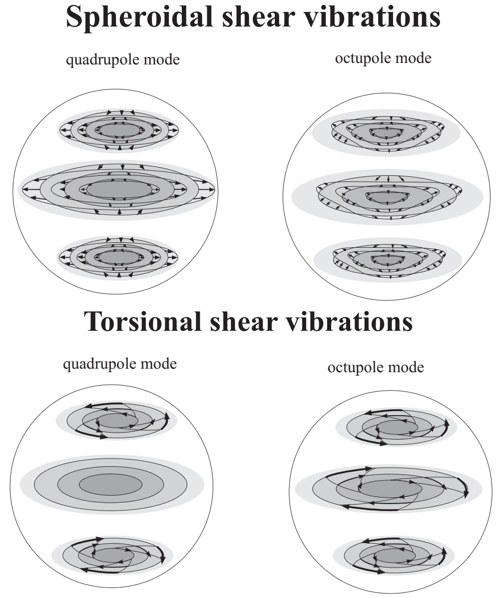

Fig. 1: Artist view of intrinsic distortions in a viscoelastic solid globe undergoing quasistatic shear oscillations in quadrupole (L=2) and octupole (L=3) overtones of spheroidal and toroidal modes

The spectral formulas for the frequency of spheroidal oscillations ω_s and the time of their viscous damping τ_s as functions of multipole degree ℓ are given by

$$\omega_s^2 = \omega_0^2\,[2(2\ell+1)(\ell-1)] \qquad \tau_s = \frac{\tau_0}{(2\ell+1)(\ell-1)} \tag{15}$$

$$\omega_0^2 = \frac{\mu}{\rho R^2} \qquad \tau_0 = \frac{\rho R^2}{\eta} \tag{16}$$

where ω_0 is the natural unit of frequency and τ_0 of the lifetime of shear vibrations.

The eigenfrequency of non-dissipative oscillations ω_t and the time of their viscous dissipation τ_t are given by

$$\omega_t^2 = \omega_0^2[(2\ell+3)(\ell-1)] \qquad \tau_t = \frac{2\tau_0}{(2\ell+3)(\ell-1)} \tag{17}$$

From spectral formulas (15) and (17) it follows: the larger multipole degree of vibration ℓ the higher frequency and the less lifetime.

3. Nuclear giant resonances in the solid globe model. By now there are quite cogent arguments showing that both the giant electric and magnetic resonances can be treated on equal solid-mechanical footing as manifestation of spheroidal (electric) and torsional (magnetic) oscillations of nuclear femtoparticle. The electromagnetic nomenclature of nuclear giant resonances owe its origin to the type of induced electromagnetic moment of fluctuating current density: the giant-resonance excitations of electric type are associated with spheroidal mode of quasistatic oscillations of irrotational field of material displacements $\mathbf{u}_s$ and those for the magnetic type with torsional mode of quasistatic differentially-rotational oscillations of toroidal field of material displacements $\mathbf{u}_t$. The multipole degree λ of electric $\mathcal{M}(E\lambda)$ and magnetic $\mathcal{M}(M\lambda)$ moments equal to multipole degree ℓ of excited spheroidal and torsional oscillatory state, respectively. The electric and magnetic multipole moment of electric current density are given by

$$\mathcal{M}(E\ell) = N_{E\ell}\int \mathbf{j}(\mathbf{r},t)\cdot\nabla r^\ell\,P_\ell\,d\mathcal{V} \quad \mathcal{M}(M\ell) = N_{M\ell}\int \mathbf{j}(\mathbf{r},t)\cdot[\mathbf{r}\times\nabla]\,r^\ell\,P_\ell\,d\mathcal{V}$$

where $\mathbf{j}(\mathbf{r},t) = \rho_e\, \dot{\mathbf{u}}(\mathbf{r},t)$ with $\rho_e = (Z/A)n$ is the charge density and n being the nucleon density. $N_{E\ell}$ and $N_{M\ell}$ are well defined constants. Note, the quasistatic oscillations of irrotational field of material displacements in spheroidal mode $\mathbf{u} = \mathbf{u}_s = \mathbf{a}_s(\mathbf{r})\,\alpha(t)$ result in excitations of vibrational states with non-zero electric multipole moment, whereas magnetic multipole moment for this kind of oscillations is zero. The opposite situation takes place for quasistatic oscillations of differentially-rotational displacements $\mathbf{u} = \mathbf{u}_t = \mathbf{a}_t(\mathbf{r})\,\alpha(t)$ in torsional mode which lead to vibrational excited states with non-zero magnetic multipole moment, while the electric multipole moment is zero. The computational formulas for the total excitation strength (probability) of giant electric and magnetic resonance are given by $B(E\ell) = (2\ell+1) < |\mathcal{M}(E\ell)|^2 >$ and $B(M\ell) = (2\ell+1)$ $< |\mathcal{M}(M\ell)|^2 >$, respectively, where bracket stands for time average.

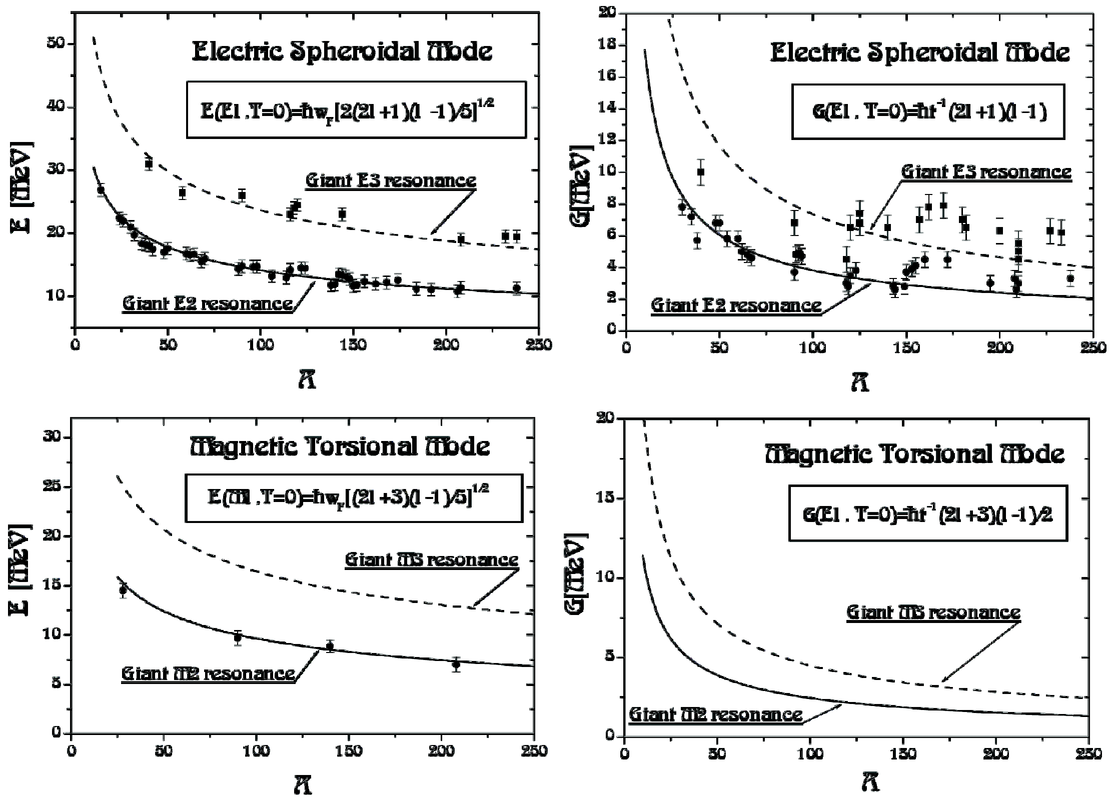

Fig. 2: The energy $E = \hbar\omega$ and width $\Gamma = \hbar\tau^{-1}$ of isoscalar electric and magnetic resonances, interpreted from equal footing as manifestation of spheroidal (electric) and torsional (magnetic) quasistatic oscillations whose frequency and lifetime is given by (15) and (17), respectively. The multipole degree λ of excited electromagnetic moment of the electric current density equal to multipole degree of oscillations $\lambda = \ell \geq 2$, and $\omega_F = v_F/R$, as pointed out in the text

Fig. 2 shows how the spectral formulas (15) and (17) of the nuclear solid globe model can be utilized to extract numerical estimates for the shear modulus and shear viscosity of nuclear matter from the experimental data of nuclear physics. The nuclear density $\rho = 2.8\,10^{14}$ g cm^{-3} and the nucleus radius $R = r_0\, A^{1/3}$ ($r_0 = 1.2\,10^{-13}$ cm) are well defined quantities. So, by fitting the data on general trends in the energy centroids and spread widths of isoscalar giant resonances with aid of standard quantum mechanical equations

$$E = \hbar\omega \qquad \Gamma = \hbar/\tau$$

we can extract the values of μ and η. This procedure leads to the following estimates

$$\eta \simeq 3 \cdot 10^{11} \text{ dyn sec cm}^{-2} \qquad \mu \simeq 10^{11} \text{ dyn cm}^{-2}$$

With the obtained from giant resonances value of shear viscosity η, one finds that the time of viscous damping of shear oscillations, $\tau_0 = \rho R^2/\eta$, in the object of nuclear density ρ and radius $R \sim 10^6$ cm, typical of neutron stars, is evaluated as $\tau_0 \sim 10^7 - 10^8$ years.

4 The solid globe models of astrophysical interest. In a highly idealized model (which is due to Landau) a neutron star is thought of as homogeneous self-gravitating mass of the degenerate neutron matter of normal nuclear density whose pressure of Fermi-degeneracy opposes the pressure of gravitational contraction under the action of its own weight. The state of gravitational equilibrium is determined by coupled equations for the potential of self-gravity and equation for the pressure

$$\nabla^2\Phi(r) = -4\pi G\rho(r) \qquad \nabla P(r) = \rho(r)\,\nabla\Phi(r). \tag{18}$$

The solution of these equations (with standard boundary conditions of gravitostatics for the potential on the globe surface, $\Phi_{in} = \Phi_{out}|_{r=R}$ and $\nabla_r\Phi_{in} = \nabla_r\Phi_{out}|_{r=R}$, and the condition of stress-free surface for the pressure, $P_{r=R} = 0$) , should be used. leads to the profile of pressure inside the star is given by $P(r) = P_c[1-(r/R)^2]$, where $P_c = (2\pi/3)G\rho^2 R^2$ is the pressure in the star center which must be equal to that for degenerate Fermi-gas of non-relativistic neutrons $P_F = K\rho^{5/3}$. This last condition, that is $P_c = P_F$ leads to canonical estimates of radius $R \approx 12$ km and mass $M = (4\pi/3)\rho R^3 \approx 1.3\, M_\odot$ of neutron stars.

From quantum-mechanical side, the fact that the pressure in the star counterbalancing Newtonian self-gravity is determined by the degeneracy pressure of Fermi-gas of independent neutron-like quasiparticles means that the single-particle states of incessant quantum-wave Fermi motion of an individual quasiparticle in the mean field of self-gravity are described by Schrödinger equation coupled with Poisson equation for the potential of the mean gravitational field $\Phi(r)$ in the star

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m_n}\nabla^2 + U(r)\right]\psi \qquad U(r) = -m_n\Phi(r)$$
$$\nabla^2\Phi(r) = -4\pi G\rho(r) \qquad \rho = \frac{m_n}{3\pi^2}\,k_F^3 = 2.8\times 10^{14}\,\text{g cm}^{-3} \tag{19}$$

where U is the potential energy of the in-medium neutron quasiparticle in its unceasing Fermi-motion whose collision free character is provided by Pauli exclusion principle; ρ stands for the uniform density in the star bulk equal to the normal nuclear density and m_n is the effective mass of the neutron quasiparticle. The solution to (19) having the form $\Phi(r) = (2\pi/3)\,G\rho\,(3R^2 - r^2)$ suggests that the potential energy U can be represented in the well-familiar form of spherical harmonic oscillator

$$U = -m_n\,\Phi(r) = -U_G^0 + \frac{m\,\omega_G^2\,r^2}{2} \qquad U_G^0 = 2\pi\,m\,\rho\,G\,R^2 \qquad \omega_G^2 = \frac{4\pi}{3}\,G\rho \tag{20}$$

where U_G^0 is the depth of spherical gravitational trap and ω_G is the characteristic unit of frequency of gravitational vibrations. The stationary states of single-particle Fermi-motion of individual neutron in the mean field of self-gravity are described by Hamiltonian

$$H\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{m\,\omega_G^2\,r^2}{2}\right]\psi = \epsilon\psi \quad \epsilon = E + U_G \tag{21}$$

whose spectrum of energies, accounted from the bottom of potential well, is well-known $\epsilon_N = \epsilon_0(N + 3/2)$ where $\epsilon_0 = \hbar\omega_G \approx 10^{-11}$ eV is the energy of zero-point oscillations which is the measure of the energy distance between single-particle states of neutron quasiparticles in the potential of self-gravity of neutron star (note the average distance between discrete states of degenerate electrons in terrestrial solids such as metals and semiconductors $\Delta \sim 10^{-18} - 10^{-20}$ eV). Here $N = 2\ell + n$ is the shell's quantum number of harmonic oscillator, n and ℓ are the principle and orbital quantum numbers of single-particle orbits, respectively. This shows that the shell-ordered clusterization of single-particle energies of neutron quasiparticles in the potential of mean gravitational field of the neutron star model has common features with that for nucleons in the mean field potential of the nuclear shell model having different physical origin.

In the star of homogeneous density the effect of gravity results in inhomogeneous pressure which has one and the same dimension with shear modulus. Bearing this in mind it is assumed that the shear modulus profile $\mu = \mu(r)$ and the shear modulus profile $\eta = \eta(r)$ are identical to that for the pressure profile $P(r)$. The model of homogeneous solid star with

$$\rho = constant \qquad \mu(r) = \mu_c \left[1 - \left(\frac{r}{R}\right)^2\right] \quad \mu_c = P_c = (2\pi/3)G\rho^2 R^2. \tag{22}$$

leads to the following estimates for the eigenfrequencies of global non-radial spheroidal and torsional quasistatic oscillations

$$\omega_s^2 = 2\omega_G^2(\ell - 1), \qquad \omega_t^2 = \omega_G^2(\ell - 1). \tag{23}$$

For a solid star – self-gravitating mass of a viscoelastic solid, the last spectral formulas seems to have the same physical meaning as Kelvin's formula does

$$\omega_f^2 = \omega_G^2 \frac{2\ell(\ell - 1)}{2\ell + 1}. \tag{24}$$

Fig. 3 shows that periods of background elastic pulsations of a homogeneous neutron star model computed with use of derived spectral formulas with the value of shear modulus extracted from data on nuclear giant resonances coincides with the timing of microspikes in the windows between the main pulses. Taking into account the above inference regarding the damping time of global quasistatic elastic vibrations of pulsars by viscosity of neutron star matter one can conclude that these microspikes owe its origin to non-radial elastodynamic (EDM) spheroidal and torsional pulsations triggered by neutron star quakes.

Summary. An understanding dynamical laws governing macroscopic motions of degenerate nucleonic material – the continuum mechanics of nuclear matter – is important for developing interconnected view of the nuclear physics and pulsar astrophysics. In this work the argument have been presented that elastodynamic approach to the continuum mechanics of nuclear matter provides proper account of macroscopic motions of degenerate Fermi-matter constituting interior of the nuclear material objects, atomic nuclei and neutron stars. In particular, it is shown that in addition to the similarity in nucleon-dominated composition and the fact of common genetic origin of nuclei and neutron stars rooted in supernovae of second type, they possesses identical orderly-organized intrinsic structure showing that degenerate nucleon Fermi-matter of nuclei and neutron

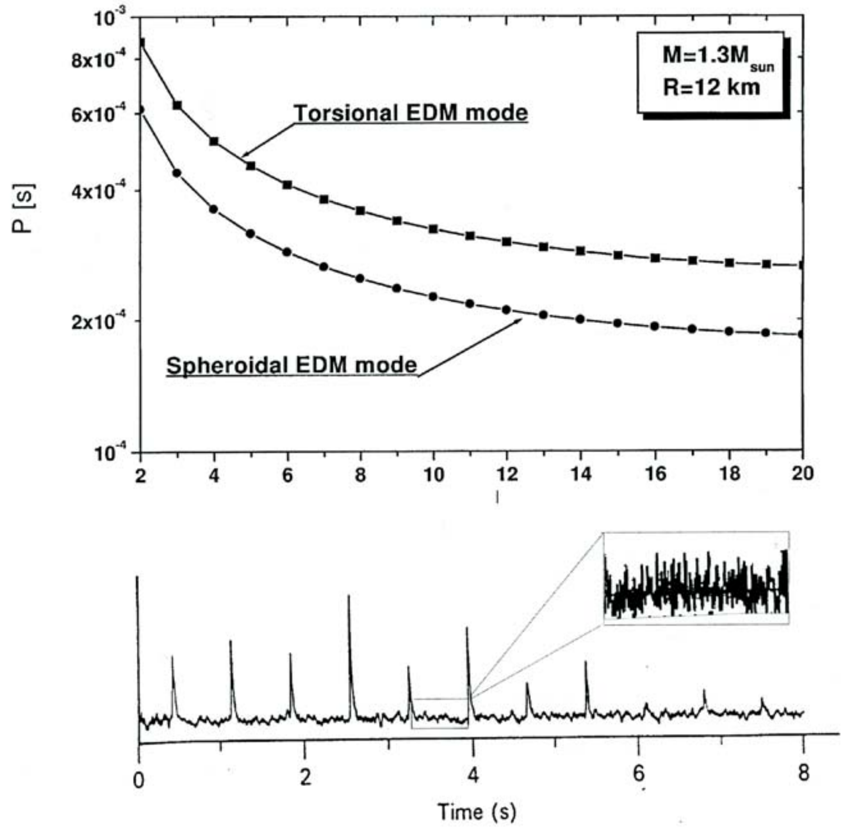

Fig. 3: Periods P against multipole degree L for spheroidal and torsional elastodynamic (EDM) modes of background non-radial pulsations of neutron star manifested by microspikes of millisecond duration in the windows of the rotation driven main pulse train

stars possesses properties of solid-mechanical viscoelasticity and can be treated therefore as elastically deformable Fermi-solid. Following this line of argument, the regime of quasistatic shear vibrations of viscoelastic solid globe has been investigated in some details, the model lying at the base of macroscopic treatment of nuclear giant resonances and data on seismic vibrations of neutron stars. It is shown how the predictions of developed theory can be utilized to gain important information about transport coefficients of nuclear matter. The practical usefulness of considered mathematical models and obtained analytic estimates is that they are quite general and can be applied to different spherical systems whose behavior is supposedly governed by equations by solid-mechanics. All details and references the interested reader can find in [1].

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