Super Energy Tensor and Dominant Energy Property: an Analysis in the Bianchi Type I Universe

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It is well-known that in the framework of general relativity the concept of local energy density is meaningless for a gravitational field. The endless search for a covariant version of gravitational energy has led to introduction of the notion of a super-energy tensor constructed with the curvature tensor $R_{\mu\nu\alpha\beta}$. The super energy tensor widely known as Bel-Robinson tensor has the following symmetry properties:

$$
\begin{aligned}
B_{\mu\nu\alpha\beta} &= B_{\nu\mu\alpha\beta}, && \text{(1a)}\\
B_{\mu\nu\alpha\beta} &= B_{\mu\nu\beta\alpha}, && \text{(1b)}\\
B_{\mu\nu\alpha\beta} &= B_{\alpha\beta\mu\nu}. && \text{(1c)}
\end{aligned}
$$

In analogy with the energy-momentum tensor of an electromagnetic field

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} + *F_{\mu\alpha} * F_{\nu}^{\alpha}, \tag{2}$$

different author defines BR in different way [1]. Here we mention only three widely met in literature.

$$B_{\mu\nu\alpha\beta} = R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}R_{\rho\nu\sigma\beta} + *R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} * R_{\rho\nu\sigma\beta}, \tag{3}$$

Here the dual curvature is $*R^{\mu\nu}{}_{\lambda\sigma} \equiv (1/2)\,\epsilon^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\sigma}$. It should be noted that this definition is adequate only in 4 dimensions and in vacuum. Since this definition imposes some restrictions on the [1]. Otherwise, this tensor cannot satisfy the dominant energy property and therefore this expression should not be used in other dimensions or in non-Ricci-flat spacetime. The restriction that arises above is due to the fact that in defining the BR tensor we used the dual term with the duality operator acting on the left pair only. To avoid this restrictions the BR tensor can be defined by

$$
\begin{aligned}
2B_{\mu\nu\alpha\beta} &= R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}R_{\rho\nu\sigma\beta} + *R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} * R_{\rho\nu\sigma\beta}\\
&+ R*^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}R*_{\rho\nu\sigma\beta} + *R*^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}*R*_{\rho\nu\sigma\beta},
\end{aligned}
\tag{4}
$$

where the duality operator acts on the left or on the right pair of indices according to its position. But the BR tensor defined in this way is not trace-free and is not completely symmetric. It is achieved if and only if the manifold is Ricci flat, i.e., $R_{ij} = 0$. Since for the BI universe we have non-trivial components of Ricci tensor, we give an alternative definition of BR where it is totally symmetric and trace-free. This leads to another definition of BR

$$B_{\mu\nu\alpha\beta} = C^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}C_{\rho\nu\sigma\beta} + *C^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} * C_{\rho\nu\sigma\beta}. \tag{5}$$

It can be shown that this BR is totally symmetric, i.e.,

$$B_{ijkl} = B_{(ijkl)},$$

Moreover, taking into account the symmetry property of Weyl tensor one can easily find that the BR defined through Weyl tensor is trace-free, i.e.,

$$g^{jl} B_{ijkl} \equiv 0. \tag{6}$$

Let us now construct the BR which meets the dominant super energy energy property (DESP). This property was formulated by Senovilla [2] in analogy with the dominant energy condition in Hawking-Penrose theorem and reads as

Theorem. *A rank-s tensor* $T_{\mu_1 \ldots \mu_s}$, *is said to satisfy the DSEP if*

$$T_{\mu_1 \ldots \mu_s} k_1^{\mu_1} \ldots k_s^{\mu_s} \geq 0 \tag{7}$$

for any future-pointing causal vectors $k_1^{\mu_1} \ldots k_s^{\mu_s}$.
To justify its name, the dominant DSEP obeys the following Lemma.

Lemma. *If a tensor* $T_{\mu_1 \ldots \mu_s}$ *satisfies the* DSEP, *then*

$$T_{0 \ldots 0} \geq |T_{\mu_1 \ldots \mu_s}|, \quad \forall \mu_1, ..., \mu_s = 0, ..., n-1 \tag{8}$$

in any orthonormal basis $\{\vec{e_\nu}\}$.

It was also established in [3], that any tensor satisfying the DEP possesses the following property:

Property. $T_{\mu_1 \ldots \mu_s}$ *satisfies DEP if and only if*

$$T_{\mu_1 \ldots \mu_s} l_1^{\mu_1} \ldots l_s^{\mu_s} \geq 0 \tag{9}$$

for any set $l_1^{\mu_1} \ldots l_s^{\mu_s}$ *of future-pointing null vectors.*

Let us now back to the 4-rank BR tensor and to check the DSEP for it. Since BR defined as (5) is a completely symmetric, trace-free 4-rank tensor, then it satisfies the DEP [4]. Therefore from the foregoing theorem, lemma and property for B_{ijkl} we can write:

$$B_{abcd} k_1^a k_2^b k_3^c k_4^d \geq 0,$$
$$B_{0 \ldots 0} \geq |B_{abcd}|, \quad \forall a, b, c, d = 0, 1, 2, 3 \tag{10}$$

and

$$B_{abcd} l_1^a l_2^b l_3^c l_4^d \geq 0. \tag{11}$$

Note that in order the BR to meet DESP stated above, one should define the BR in terms of Weyl tensor as in (5) and essentially in an orthonormal basis. We show this within the scope of a Bianchi type-I spacetime given by

$$ds^2 = dt^2 - a_1(t)^2 dx^2 - a_2(t)^2 dy^2 - a_3(t)^2 dz^2. \tag{12}$$

Indeed, in an orthonormal basis the components of the Weyl tensor obey the following relation:

$$C_{0i0i} = -C_{jkjk}, \quad i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k. \tag{13}$$

In view of (13) one finds the following expressions for the nontrivial components of the BR tensor [5, 6, 7]

$$
\begin{aligned}
B_{0000} &= \Big(C_{jkjk}\Big)^2 + \Big(C_{kiki}\Big)^2 + \Big(C_{ijij}\Big)^2 && \text{(14a)}\\
B_{iiii} &= B_{0000}, && \text{(14b)}\\
B_{ijij} &= 2C_{0i0i}C_{0j0j} && \text{(14c)}\\
B_{0k0k} &= -B_{ijij}. \quad i,j,k = 1,2,3, \quad i \neq j \neq k. && \text{(14d)}
\end{aligned}
$$

Thus in connection with the above Lemma relative to DSEP, Eq. (10) is fulfilled without any restrictions on the metric functions.

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