Anisotropic Cosmological Models with Dark Energy

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1 Introduction

Recent observations suggest that the universe is spatially flat and undergoing a period of accelerated expansion. In order to explain this accelerated mode of expansion of the present day Universe, cosmologists introduced different kind source fields:

- **Λ term:** First introduced by Einstein in 1917. After the discovery of late time acceleration it was resurrected.
- **Quintessence:** Most popular DE model with the equation of state $w = p_{\rm DE}/\varepsilon_{\rm DE}$ with $w \in [-1,\, -1/3]$.
- **Chaplygin gas:** To unify two completely different physical concept such as dark energy (DE) and dark matter (DM), an exotic EoS was suggested $p_{\rm DE} = A/\varepsilon_{\rm DE}$ with $A > 0$.
- **Phantom DE:** In this case $w < -1$. Energy density increases as the Universe expands and leads to an future singularity known as a big rip. The universe becomes infinitely large within a finite time.
- **Oscillating DE:** This model was suggested to avoid eternal acceleration. Most popular is the cyclic Universe which begins with a big bang and ends in a big crunch only to emerge in a big bang again.
- **Spinor fields:** Introduction of nonlinear spinor field leads to the rapid expansion of the universe. Recently it was shown that the spinor field can be viewed as an alternative to DE.
- **Models with interaction between DE and dark matter:**
- **Scalar-tensor DE models:**
- **Tachyon models:**
- **and many others**

In this report in the framework of BI cosmological model we study the evolution of the universe in presence of different time of source fields able to explain the late time acceleration.

2 BI Universe: a brief review

An anisotropic BI model is given by the metric

$$ds^2 = dt^2 - a_1^2 dx_1^2 - a_2^2 dx_2^2 - a_3^2 dx_3^2, \tag{1}$$

where a_i are the functions of t only and the speed of light is taken to be unity. We also define

$$\tau = a_1 a_2 a_3. \tag{2}$$

Einstein's gravitational field equation corresponding to the BI spacetime we write in the form:

$$\frac{\ddot a_i}{a_i} + \frac{\ddot a_j}{a_j} + \frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j} = \kappa T_k^k, \quad i \neq j \neq k = 1,2,3 \tag{3a}$$

$$\frac{\dot a_1}{a_1}\frac{\dot a_2}{a_2} + \frac{\dot a_2}{a_2}\frac{\dot a_3}{a_3} + \frac{\dot a_3}{a_3}\frac{\dot a_1}{a_1} = \kappa T_0^0. \tag{3b}$$

Here T_μ^ν is the energy-momentum tensor of the source fields (DE, spinor and scalar fields and the perfect fluid etc.). In the models studied here we have $T_1^1 = T_2^2 = T_3^3$. Under this from Eqns. (3a) we find

$$a_i(t) = A_i [\tau(t)]^{1/3} \exp\bigl[X_i \int [\tau(t')]^{-1} dt'\bigr], \tag{4}$$

with the integration constants A_i and X_i obeying the conditions $A_1 A_2 A_3 = 1$ and $X_1 + X_2 + X_3 = 0$.

As it was mentioned above, in order to give a more realistic description of the early day Universe we need to consider the anisotropic cosmological models such as Bianchi type-I (BI). Since the modern day Universe is wonderfully isotropic, we have to find out how and when the initially anisotropic spacetime evolves into an isotropic one. There exists a number of isotropization criteria in literature. Here are the most common two

$$\mathcal{A} = \frac{1}{3}\sum_{i=1}^{3}\frac{H_i^2}{H^2} - 1, \quad \Sigma^2 = \frac{1}{2}\mathcal{A}H^2. \tag{5}$$

Here $\mathcal{A}$ and Σ^2 are the mean anisotropy parameter and shear parameter, respectively. $H_i = \dot{a}_i/a_i$ are the directional Hubble parameters and $H = \dot{a}/a$ is the mean Hubble parameter, with $a(t) = \tau^{1/3}$ being the mean scale factor. Isotropization means that at large physical times, when the volume scale τ tends to infinity, the three scale factors $a_i(t)$ grow at the same rate. Therefore, we will say that a model is isotropizing if

$$a_i/a \to \text{const} > 0 \qquad \text{as} \qquad \tau \to \infty. \tag{6}$$

From (4) follows $a_i/a \to A_i = \text{const.}$ as $\tau \to \infty$. Recalling that the isotropic FRW model has $a_1(t) = a_2(t) = a_3(t) = a(t)$ we assume that $A_1 = A_2 = A_3 = 1$. Note that by re-scaling some coordinates we can come to $a_i/a \to 1$ and the metric will become manifestly isotropic at large t. Moreover, the isotropic nature of the present Universe leads to the fact that the three other constants X_i should be close to zero as well, i.e., $|X_i| << 1$, $(i = 1, 2, 3)$, so that $X_i \int [\tau(t)]^{-1} dt \to 0$ for $t < \infty$ (for $\tau(t) = t^n$ with $n > 1$ the integral tends to zero as $t \to \infty$ for any X_i). The rapid growth of the Universe due to the introduction of the nonlinear spinor field to the system results in the earlier isotropization.

We define the generalized deceleration parameter as

$$d = \frac{d}{dt}\Big(\frac{1}{3H}\Big) - 1 = -\frac{\tau\ddot{\tau}}{\dot{\tau}^2}. \tag{7}$$

Thus we see that the volume scale plays special role in BI universe as most of the expressions are either obtained or defined in terms of τ. So let us first write the equation for τ. Taking into account that $T_1^1 = T_2^2 = T_3^3$ after a little manipulation from (3) one finds the equation for τ which is indeed the acceleration equation and has the following general form:

$$\frac{\ddot{\tau}}{\tau} = \frac{3}{2}\kappa\Big(T_1^1 + T_0^0\Big). \tag{8}$$

On the other hand from the Bianchi identity $G_{\mu;\nu}^{\nu} = 0$ we have

$$\dot{T}_0^0 = -\frac{\dot{\tau}}{\tau}\big(T_0^0 - T_1^1\big). \tag{9}$$

In order to resolve the Eqns. (8) and (9) we need one more condition, which is given by the equations of state (EoS). It should be noted that the dark energy is supposed to interact with itself only and it is minimally coupled to the gravitational field. As a result the evolution equation for the energy density decouples from that of the perfect fluid, and from Eq. (9) we obtain two balance equations: one for perfect fluid and the other for dark energy. In case of a spinor and/or scalar field corresponding terms in (9) cancels each other thanks to the field equations leaving the equation for perfect fluid only. In what follows we study the evolution of the BI universe for some concrete type of source fields. It can be shown that for the cases considered here, the right hand side of (8) is a function of τ, hence it allows first integral. Denoting RHS as $\mathcal{F}(\tau)$ the solution can be given as

$$\dot{\tau} = \sqrt{2[C - \mathcal{U}(\tau)]}, \tag{10}$$

where $E = -\int \mathcal{F}(\tau) d\tau$ can be view as some potential and the integration constant C as energy level.

3 Universe as a binary mixture of perfect fluid and dark energy

In this section we study the evolution of the universe for the given type of dark energy in presence of perfect fluid.

3.1 Models with Λ term

Let us consider a BI model with a Λ term in presence of a perfect fluid. As a perfect fluid we choose an usual one obeying barotropic EoS or one with Van der Waals EoS. This case was studied in details in [1].

3.1.1 Perfect fluid with barotropic EoS

Perfect fluid in this case obeys

$$p_{\rm pf} = \zeta\,\varepsilon_{\rm pf}, \quad \zeta \in [0,\,1]. \tag{11}$$

From (9) one now finds

$$\varepsilon_{\rm pf} = \varepsilon_0/\tau^{(1+\zeta)}, \qquad p_{\rm pf} = \varepsilon_0\zeta/\tau^{(1+\zeta)}, \quad \varepsilon_0 = {\rm const.} \tag{12}$$

In the Figs. 1 we illustrated the potential in case of a negative Λ. As one sees in case of stiff matter this potential allows only non-periodic solution. Evolution of τ corresponding to the negative Λ is given in Figs. 2 and 3. As one sees, depending on the value of energy level (C) we have either oscillatory or non-periodic mode of expansion. The oscillatory solutions are regular everywhere, while the non-periodic ones ends in a big crunch (future singularity). In Fig. 4 we illustrated the evolution of τ corresponding to the positive Λ. The anisotropy in this case dies away quicker

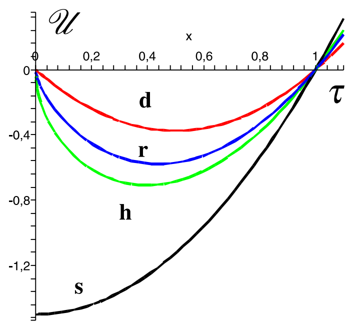

Fig. 1: View of the potential $\mathcal{U}(\tau)$

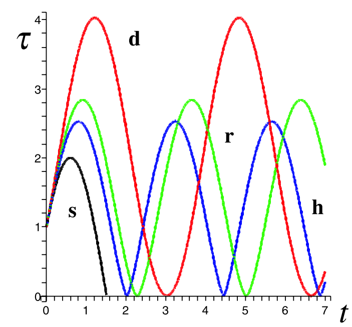

Fig. 2: Evolution of volume scale τ with a negative Λ and $C = -0.1$

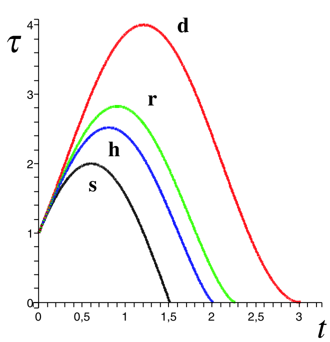

Fig. 3: Evolution of volume scale τ with a negative Λ and $C = 0$

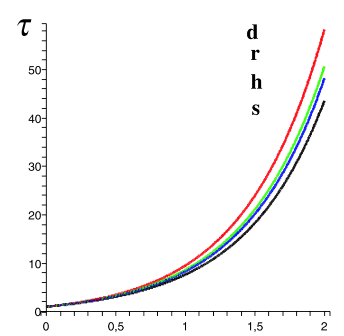

Fig. 4: Evolution of the Universe with a positive positive Λ

3.1.2 Perfect fluid with Van der Waals EoS

The pressure of the van der Waals fluid $p_{\rm w}$ is related to its energy density $\varepsilon_{\rm w}$ by

$$p_{\rm w} = \frac{8W\varepsilon_{\rm w}}{3-\varepsilon_{\rm w}} - 3\varepsilon_{\rm w}^2. \tag{13}$$

This case was thoroughly studied in [1]. Here we present main results. In Figs. 5 and 6 energy density and pressure of a Van der Waals fluid are illustrated for negative and nonnegative Λ, respectively. In Fig. 7 the evolution of τ is illustrated. Independent to the sign of Λ the model provides provides with rapidly expanding Universe. Fig. 8 demonstrates the acceleration of BI universe filled with Van der Waals fluid for different Λ. As one sees (Fig. 6), Van der Waals fluid possesses negative pressure at the initial time, hence can be exploited to explain inflation at the

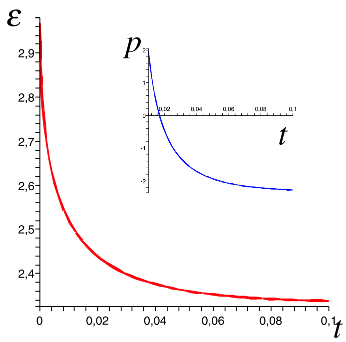

Fig. 5: View of energy density ε and pressure p of a Van der Waals fluid with a negative Λ

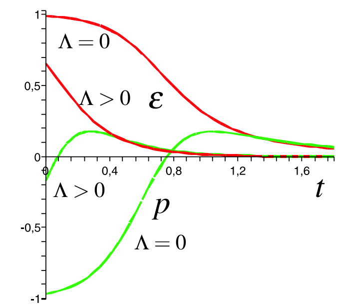

Fig. 6: View of energy density ε and pressure p of a Van der Waals fluid with $\Lambda \geq 0$

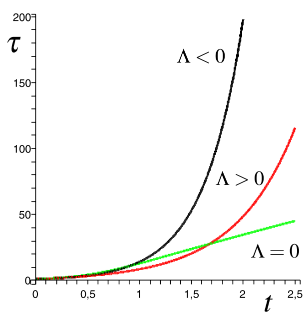

Fig. 7: Evolution of τ with the BI Universe filled with Van der Waals fluid

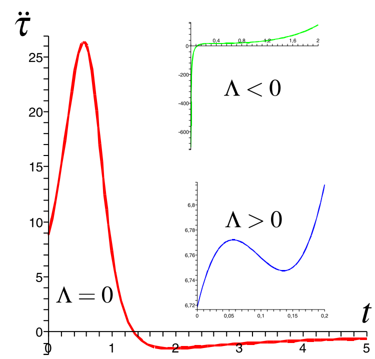

Fig. 8: Acceleration of a BI Universe filled with a Van der Waals fluid for different Λ

3.2 Models with quintessence

Model with quintessence and chaplygin gas in BI unverse was studied in [2]. As it was mentioned earlier quintessence is given by the EoS

$$p_{\mathrm{q}} = w_{\mathrm{q}} \varepsilon_{\mathrm{q}}, \tag{14}$$

where the constant w_{q} varies between -1 and zero, i.e., $w_{\mathrm{q}} \in [-1, 0]$. The perfect fluid is given by (11). It should be noted that $w_{\mathrm{q}} = -1$ corresponds to a Λ term, while $w_{\mathrm{q}} < -1$ corresponds to a phantom DE. Energy density of perfect fluid is related to τ (12). In account of (14) one finds

$$\varepsilon_{\mathrm{q}} = \varepsilon_{0\mathrm{q}}/\tau^{(1+w_{\mathrm{q}})}, \qquad p_{\mathrm{q}} = w_{\mathrm{q}} \varepsilon_{0\mathrm{q}}/\tau^{(1+w_{\mathrm{q}})}, \tag{15}$$

with $\varepsilon_{0\mathrm{q}}$ being some integration constant.

In Figs. 9 and 10 we illustrated the potential and evolution of τ in case of a quintessence (q), phantom (ph), perfect fluid (pf) and Chaplygin gas (ch), respectively. As one sees, in case of a phantom the model provides big rip (Universe becomes infinitely large within finite time).

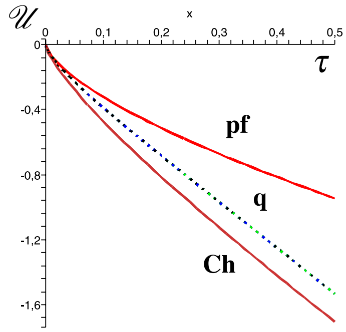

Fig. 9: View of potentials when the Universe is filled with perfect fluid, perfect fluid plus quintessence and perfect fluid plus Chaplygin gas, respectively

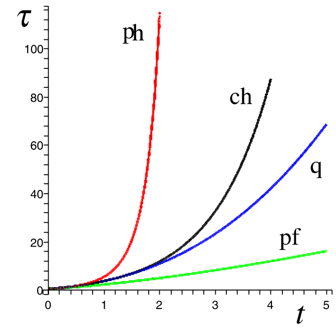

Fig. 10: Evolution of the BI Universe corresponding to the potentials illustrated in Fig. 9

3.3 Case with Chaplygin gas

Let us now consider the case when the dark energy is represented by a Chaplygin gas. We have already mentioned that the Chaplygin gas was suggested as an alternative model of dark energy with some exotic equation of state, namely

$$p_{\rm c} = -A/\varepsilon_{\rm c}, \tag{16}$$

with A being a positive constant. In view of the Eq. (16) one now obtains

$$\varepsilon_{\rm c} = \sqrt{\varepsilon_{0{\rm c}}/\tau^2 + A}, \qquad p_{\rm c} = -A/\sqrt{\varepsilon_{0{\rm c}}/\tau^2 + A}, \tag{17}$$

with $\varepsilon_{0{\rm c}}$ being some integration constant.

Proceeding analogously as in previous case for τ we now have

$$\ddot{\tau} = \frac{3\kappa}{2}\Bigl(\frac{(1-\zeta)\varepsilon_0}{\tau^\zeta} + \sqrt{\varepsilon_{0{\rm c}} + A\tau^2} + A/\sqrt{\varepsilon_{0{\rm c}} + A\tau^2}\Bigr). \tag{18}$$

The corresponding solution in quadrature now has the forms:

$$\int \frac{d\tau}{\sqrt{C_1 + 3\kappa\bigl(\varepsilon_0\tau^{(1-\zeta)} + \sqrt{\varepsilon_{0{\rm c}}\tau^2 + A\tau^4}\bigr)}} = t, \tag{19}$$

where the second integration constant has been taken to be zero.

3.4 Case with modified quintessence

In order to get rid of the eternal acceleration different authors suggest different models. A modified quintessence model able to give a regular mode of expansion was proposed in [3] with the following EoS

$$p_{\rm mq} = -w(\varepsilon_{\rm mq} - \varepsilon_{\rm cr}), \tag{20}$$

where the constant $w \in [0,\,1)$. Here $\varepsilon_{\rm cr}$ some critical energy density. Setting $\varepsilon_{\rm cr} = 0$ one obtains ordinary quintessence. It is well known that as the Universe expands the (dark) energy density decreases. As a result, being a linear negative function of energy density, the corresponding pressure begins to increase. In case of an ordinary quintessence the pressure is always negative, but for a modified quintessence as soon as $\varepsilon_{\rm q}$ becomes less than the critical one, the pressure becomes positive. In Fig. 11 evolution of energy density and pressure is demonstrated. In Fig. 12 we show the acceleration for different source fields. Here "rad", "quint" and "quint-m" stand for radiation, a mixture of radiation and an ordinary quintessence and a mixture of radiation and modified quintessence, respectively.

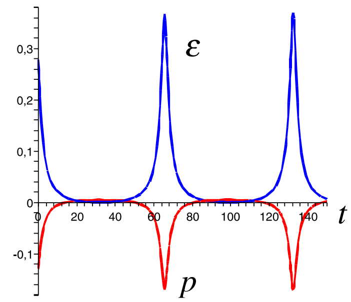

Fig. 11: View of energy density and pressure when BI Universe experiences oscillation

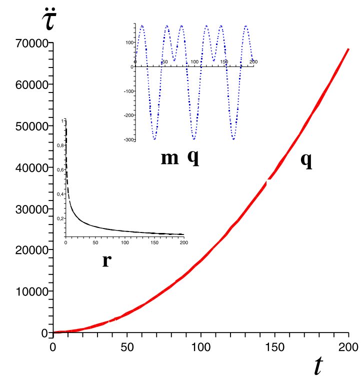

Fig. 12: View of the acceleration for different source fields

4 Spinor field as an alternative to DE

In this section we study the role of spinor field in the evolution of a BI universe and examine if a spinor field can be exploited to explain the early inflation and late time acceleration of the Universe. This study was thoroughly carried out in [4, 5, 6]. We consider the case when the spinor field nonlinearity is given by a self action (corresponds to the second term in energy density and first term in pressure given in (21), respectively) or by an interaction with a scalar field (corresponds to the third term in energy density and second term in pressure, respectively). Here the nonlinearity is taken to be a power law of the invariants of the bilinear spinor form. For energy density and pressure in account that of the perfect fluid we now have

$$\begin{aligned} T_0^0 &= \frac{m}{\tau} - \frac{\lambda}{\tau^q} + \frac{\tau^{r-2}}{2(\lambda_1 + \tau^r)} + \frac{1}{\tau^{1+\zeta}}, \\ T_1^1 &= \frac{(q-1)\lambda}{\tau^q} - \frac{[(1-r)\lambda_1 + \tau^r]\tau^{r-2}}{2(\lambda_1+\tau^r)^2} - \frac{\zeta}{\tau^{1+\zeta}}. \end{aligned} \tag{21}$$

Taking into account that T_0^0 and T_1^1 are the functions of τ only, the Eq. (8) can now be presented as

$$\ddot{\tau} = \mathcal{F}(q_1, \tau), \quad \mathcal{F}(q_1,\tau) = \frac{3\kappa}{2}\Big(m + \frac{\lambda(q-2)}{\tau^{q-1}} + \frac{\lambda_1 r \tau^{r-1}}{2(\lambda_1+\tau^r)^2} + \frac{1-\zeta}{\tau^\zeta}\Big), \tag{22}$$

where $q_1 = \{\kappa, m, \lambda, \lambda_1, q, r, \zeta\}$ is the set of problem parameters. Corresponding potential in this case reads:

$$\mathcal{U}(q_1,\tau) = -\frac{3}{2}\Big[\kappa\Big(m\tau - \lambda/\tau^{q-2} - \lambda_1/2(\lambda_1+\tau^r) + \tau^{1-\zeta}\Big)\Big]. \tag{23}$$

From (22) one finds $\ddot{\tau} \to (3/2)\kappa m > 0$ as $\tau \to \infty$, i.e., if $\ddot{\tau}$ is considered to be the acceleration of the BI Universe, then the massive spinor field essentially can be viewed as a source for ever lasting acceleration. As far as initial stage of expansion is concerned (here we are exclusively dealing with an expanding Universe), the positivity of the radical imposes some restriction on the value of τ, namely in case of $\lambda > 0$ and $q \geq 2$ the value of τ cannot be too close to zero at any space-time point. In this case there exists an infinitely high potential wall as $\tau \to 0$ making it impossible for any classical system to reach the point where $\tau = 0$ (Fig. 13). Thus we conclude that for some special choice of problem parameters the introduction of nonlinear spinor field given by a self-action provides singularity-free solutions.

Let us now consider the case when λ is negative. From (23) one sees, in the vicinity of $\tau = 0$ there exists a bottomless potential hole (Fig. 14). If the initial value of τ is too close to zero and the constant E is less than $\mathcal{U}_{\rm max}$ (the maximum value of the potential in presence of a self-action), the Universe will never come out of the hole.

In what follows, we present the results of numerical simulation graphically. In the figures "1","2" and "3" correspond to the case with self-action plus interaction, self-action only and interaction, respectively.

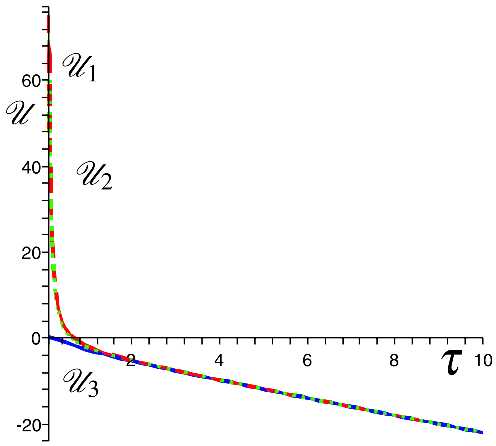

Fig. 13: View of the potential $\mathcal{U}(\tau)$ for $\lambda > 0$

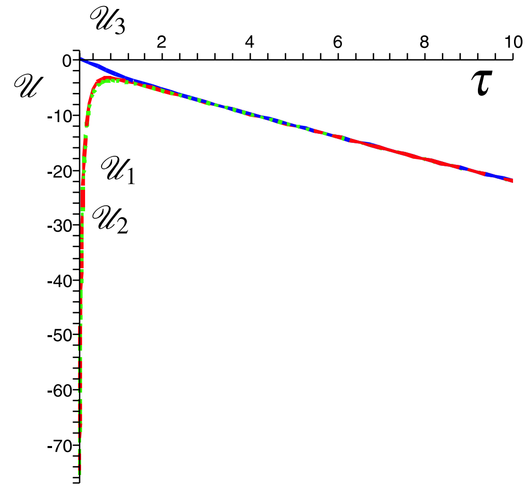

Fig. 14: View of the potential $\mathcal{U}(\tau)$ for a negative λ

In Figs. 15 and 16 we plot the corresponding energy density and pressure. In case of a positive λ the energy density is initially negative while the pressure is positive. In this case though the solution is singularity-free, the violation of dominant energy takes place. In case of a negative λ the pressure is always negative.

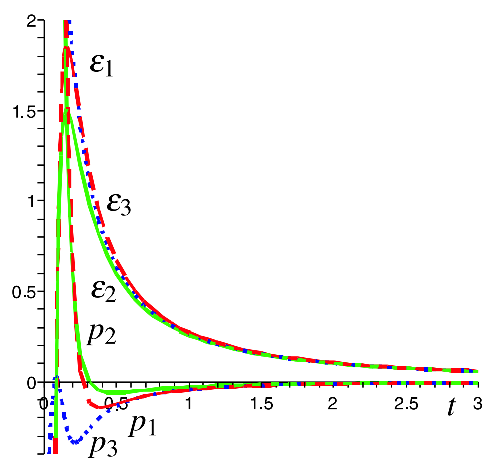

Fig. 15: Energy density and pressure corresponding to a positive λ

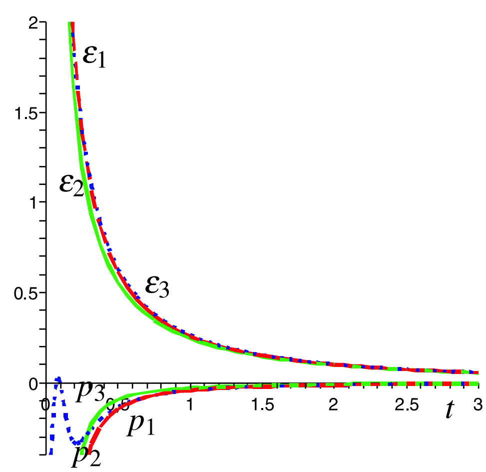

Fig. 16: Energy density and pressure in case of a negative λ

As the figures show, in case of self-action with a positive λ pressure is initially positive, but with the expansion of the Universe it becomes negative, whereas, in case of of a negative λ as well as in case of interacting fields the pressure is always negative. Recall that the dark energy (e.g. quintessence, Chaplygin gas), modelled to explain the late time acceleration of the Universe, has the negative pressure. So we argue that the models with nonlinear spinor field and interacting spinor and scalar fields to some extent can be considered as an alternative to dark energy which is able to explain the late time acceleration of the Universe.

In the Figs. 17 and 18 we illustrate the acceleration of the Universe for positive and negative λ, respectively. As one sees, in both cases we have decreasing acceleration that tends to $(3/2)\kappa m$ as $\tau \to \infty$. Depending of the choice of nonlinearity it undergoes an initial deceleration phase. It is also seen that the nonlinear term plays proactive role at the initial stage while at the later stage spinor mass is crucial for the accelerated mode of expansion. Given the fact that neutrino possesses mass and there exists a huge amount of neutrino in nature, our result indicates at neutrino as one of the possible candidates to explain late time acceleration of the Universe.

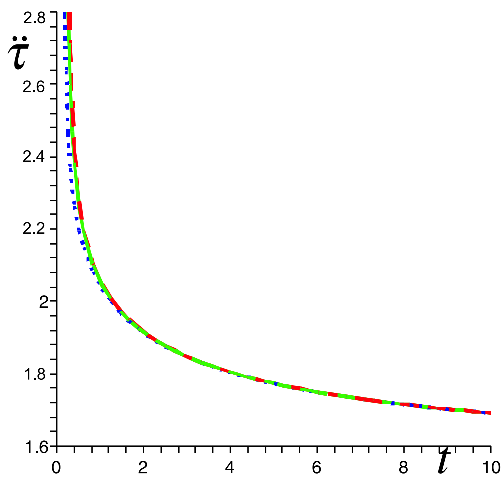

Fig. 17: Acceleration of the Universe corresponding to a positive λ

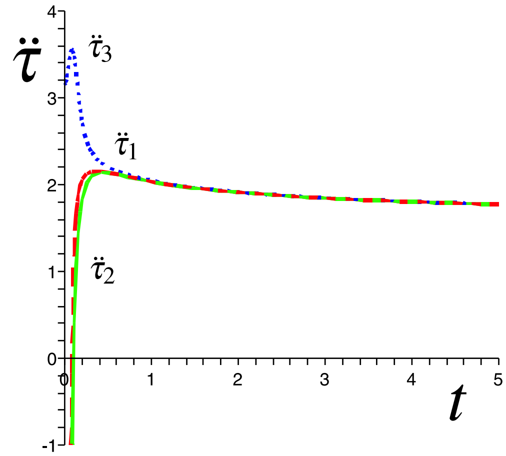

Fig. 18: Acceleration of the Universe in case of a negative λ

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