# Generalized Electrodynamics Based on Ternary Algebraic Structures

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**Abstract**

We build the new dynamics based on ternary mapping between outer- and inner- momenta. In polar representation the inner- and outer- momenta obey the Jacobi and the Weierstrass equations, correspondingly. The theory is constructed in $4D$ space with Euclidean metric which possesses an advantage to build ternary mapping three vectors onto one. This remarkable property allows one to construct a tensorial form of the evolution equations for inner- and outer- momenta. The Hamilton-Jacobi equations are derived, the analogues of Klein-Gordon and Maxwell equations for triple fields are formulated.

**Introduction**

In the present contribution we give a sketch of the theory which is natural extension of the relativistic electrodynamics. The algebraic structure of this mechanics is characterized by three-order polynomial. The mechanics deals with the triplet of energies. The earlier publications on the subject the reader may find in Refs.[1] where the following conclusions had been derived: (a) the relativistic mechanics deals with some special system which contains the pair of energies; (b) the dynamics is described by two sets of the momenta which are inter-related as coefficients and eigenvalues of the quadratic polynomial. These sets of momenta were named as outer- and inner-momenta of the particle, correspondingly. The equations for outer- and inner- momenta complement each other and together can be formulated as Heisenberg equations for a finite (four-) dimensional quantum system. This formulation serves as a heuristic starting point in order to extend the frames of the classical relativistic mechanics. As a direct extension of this scheme we build the new dynamics based on ternary mapping between outer- and inner- momenta. In polar representation the inner- and outer- momenta obey the Jacobi and the Weierstrass equations, correspondingly. The theory is constructed in $4D$ space with Euclidean metric which possesses an advantage to build ternary mapping three vectors onto one. The novelties are the cubic order Hamilton-Jacobi equations, cubic order Klein-Gordon equation and generalized Maxwell equations with triple fields.

## 1 Two sets of momenta of the relativistic particle

Let us emphasize that one of the important features of the relativistic dynamics of charged particle is that this dynamics is formed by the pair of energies. These quantities are eigenvalues of the quadratic polynomial which, if the energy is referenced from the internal energy $Mc^2$, is is reduced to the mass-shell equation. The coefficients of this polynomial form a set of outer-momenta, the eigenvalues form another set of the squared momenta denominated as inner-momenta of the relativistic particle. Thus, the dynamics can be represented by two kinds of equations corresponding to these two sets of momenta. Evolution equations in terms of outer-momenta are nothing else than the Lorentz-force equations. Equations in terms of the inner-momenta are formulated in $4D$ space (not space-time) in the basis of quaternion algebra. Equations for the outer-momenta are consequence of the equations for inner-momenta.

Consider a motion of the relativistic particle with charge $e$ in the external electromagnetic fields $\vec{E}$ and $\vec{B}$. The relativistic equations of motion with respect to the proper time $\tau$ are given by the Lorentz-force equations [2]:

$$\frac{d\vec{P}}{d\tau} = \frac{e}{mc}\vec{E}\ P_0 + \frac{e}{m}[\vec{P} \times \vec{B}]), \quad \frac{dP_0}{d\tau} = \frac{e}{mc}(\vec{E} \cdot \vec{P}), \tag{1.1}$$

$$\frac{d\vec{r}}{d\tau} = \frac{\vec{P}}{m}, \quad \frac{dt}{d\tau} = \frac{P_0}{mc}. \tag{1.2}$$

These equations imply the first integral of motion

$$P_0^2 - P^2 = M^2c^2. \tag{1.3}$$

In the case of stationary potential field, i.e. when $e\vec{E} = -\vec{\nabla}V(r)$, the equations imply the other constant of motion, the energy of the relativistic particle

$$\mathcal{E} = cP_0 + V(r). \tag{1.4}$$

Correspondence with the non-relativistic equations gives an interpretation of the constant of motion $M^2$ as a squared mass of the particle. One of the important features of the relativistic dynamics of charged particle is that this dynamics is formed by the pair of energies:

$$\mathrm{p}^2 = \frac{p^2}{2m} = cP_0 - Mc^2, \quad \mathrm{q}^2 = \frac{q^2}{2\mu} = cP_0 + Mc^2. \tag{1.5}$$

The first one in the non-relativistic limit this value is transformed into kinetic energy of the Newtonian particle. These energies form the set of eigenvalues of the quadratic polynomial

$$X^2 - 2cP_0\ X + c^2P^2 = 0, \quad P_0^2 \geq P^2 \geq 0. \tag{1.6}$$

Two solutions of Eq.(1.6) distinct with the sign of mass. The quantities $q,\ \mu$ are given in units of the energy. By using Vieta's formulae from (1.3) and (1.5) we come to the following mapping between inner and outer momenta

$$c^2P^2 = \mathrm{p}^2\mathrm{q}^2, \quad cP_0 = \frac{1}{2}(\mathrm{q}^2 + \mathrm{p}^2), \quad Mc^2 = \frac{1}{2}(\mathrm{q}^2 - \mathrm{p}^2). \tag{1.7}$$

At the rest, $P = 0,\ P_0 = mc$ and $p = 0$, the $q$-kinetic energy becomes $cP_0 + mc^2 = 2mc^2$. In the sequel, we propose define $\mu$ as the value of $q$ at the rest: $q(p = 0) = \mu$. Then the previous equality gives $\mu = 4mc^2$. The dynamic equations for the inner momenta are:

$$(a)\ \ \frac{dp}{d\tau} = e(\vec{n}\cdot\vec{E})\ \frac{q}{\mu}, \quad (b)\ \ \frac{dq}{d\tau} = e(\vec{n}\cdot\vec{E})\ \frac{p}{m}, \quad (c)\ \ \frac{d\vec{r}}{d\tau} = \vec{n}\ \frac{p}{m}\frac{q}{\mu}. \tag{1.8}$$

In the case of stationary potential field Eqs.(1.8) imply two constants of motion each of which has a form of Newtonian energy:

$$\mathcal{E}_p = \frac{p^2}{2m} + V(r), \quad \mathcal{E}_q = \frac{q^2}{2\mu} + V(r).$$

This form of the energies prompts an idea that Eqs.(1.8) admit partition into two Newtonian equations. In fact, by replacing the parameter of evolution $\tau$ by $t_p$, where $\mu dt_p = q d\tau$, or by $\rho$ satisfying $m d\rho = p d\tau$, correspondingly, equations (1.8) are reduced into

$$(a)\ \ \frac{dp}{dt_p} = -\frac{dV(r)}{dr}, \ \ \frac{dr}{dt_p} = \frac{p}{m}; \quad (b)\ \ \frac{dq}{d\rho} = -\frac{dV(r)}{dr}, \ \ \frac{dr}{d\rho} = \frac{q}{\mu}. \tag{1.9}$$

One may identify $p,\ \mathcal{E}_p$ and $q,\ \mathcal{E}_q$ with the momentum - energy of the $p$- and $q$-carriers, respectively. Dynamic equations for the coupled $p$- and $q$- carriers written with respect to unique time are given by Eqs.(1.8). So, within the scope of the present terminology, the $p$- and $q$- particles mean two states of the *relativistic particle*. These states in the experiment are observed as a particle, or an anti-particle. In the present picture one may explicitly observe a succession between relativistic and Newtonian mechanics, where $p$-particle play a role of the successor.

The present scheme can be developed in a covariant form. One of the covariant formulations can be done in spinorial form [3]. Obviously, in this case the scheme will loose its physical sense in contrary to the one dimensional case. In the sequel the spinorial formulation had been extended within the twistorial formulation where the $p, q$-sub-particles have been presented as some massless particles [4]. However, as it has been shown in Refs.[5] the full theory can be formulated only in $4D$ Euclidean space. For a physicist the $4D$-euclidean space is more attractive than the $3D$-Euclidean space. One of the advantages of the former is connection with the $SU(2)$ group and the quaternionic calculus. In fact, the basic spin of the nature, spin one-half, is described by $SU(2)$ group which closely related with $4D$-Euclidean space because the isomorphism between $SU(2) \otimes SU(2)$ and $SO(4)$ groups. Thus, the adequate mathematical tool in $4D$ space is the quaternion algebra. Define quaternions of the inner momenta by

$$\mathbf{p} = \{p_4 + (\vec{\kappa} \cdot \vec{p})\}, \ \mathbf{q} = \{q_4 + (\vec{\kappa} \cdot \vec{q})\}.$$

In the quaternionic basis the mapping (1.7) is extended as follows

$$P_0 = \frac{1}{2}(\mathbf{p}\bar{\mathbf{p}} + \mathbf{q}\bar{\mathbf{q}}), \quad M = \frac{1}{2}(\mathbf{q}\bar{\mathbf{q}} - \mathbf{p}\bar{\mathbf{p}}), \quad \mathbf{P} = \mathbf{p}\mathbf{q}. \tag{1.10}$$

In order to formulate all equations in the basis of quaternions the electromagnetic fields also should be represented in the quaternionic basis. Define two quaternions

$$\mathbf{B} = B_4 + (\vec{\kappa} \cdot \vec{B}), \ \mathbf{E} = E_4 + (\vec{\kappa} \cdot \vec{E}).$$

The motion inside electric and magnetic fields in terms of the inner-momenta are described by the following equations

$$\frac{d}{d\tau}\mathbf{p} = \frac{e}{2m}\left( \ \bar{\mathbf{q}}\mathbf{E} + [\mathbf{B}\mathbf{p} - \mathbf{p}\mathbf{B}] \ \right), \frac{d}{d\tau}\mathbf{q} = \frac{e}{2m}\left( \ \mathbf{E}\bar{\mathbf{p}} + [\mathbf{q}\mathbf{B} - \mathbf{B}\mathbf{q}] \ \right). \tag{1.11}$$

From these equations the following equations for the outer-momenta are derived

$$\frac{d}{d\tau} \ \mathbf{P} = \frac{e}{2m}\left([\mathbf{B}\mathbf{P} - \mathbf{P}\mathbf{B}] + 2\mathbf{E} \ P_0\right), \quad \frac{d}{d\tau} \ P_0 = \frac{e}{m}\Big( \ (\vec{E} \cdot \vec{P}) + E_4 P_4 \ \Big), \tag{1.12}$$

with the first integral

$$P_0^2 - \vec{P}^2 - P_4^2 = M^2. \tag{1.13}$$

That is the de-Sitter surface equation imbedded in $5D$ momentum space.

## 2 Maxwell field equations in $4D$ space

In $4D$-space in the basis of quaternion algebra the Maxwell equations is given by the following system of equations

$$\Big(\nabla_4 - (\vec{\kappa} \cdot \vec{\nabla})\Big) \mathbf{H} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = (\vec{\kappa} \cdot \vec{j}), \quad \Big(\nabla_4 + (\vec{\kappa} \cdot \vec{\nabla})\Big) \mathbf{E} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{H} = -\rho.$$

For our purpose it is necessary to formulate the Maxwell equations in $4D$-space in tensorial form. The strengths $E_k, \ B_{mn}$ are $4D$-analogues of the electric and magnetic fields. As in the case of ordinary electromagnetic fields, the electric field strength is represented by the $4D$-vector whereas the magnetic part is given by bi-vector. The Maxwell field equations are given by the following system

$$(a) \ \partial_t \ E_k - \frac{1}{2}\epsilon_{klmn}\partial_l B_{mn} = -j_k, \quad (b) \ \partial_t \ B_{mn} = -\epsilon_{mnkl}\partial_k E_l. \tag{2.1}$$

These equations have to be complemented with the equations meaning an existence of the electric charge and an absence of the magnetic charges:

$$(a)\ \ \partial^l E_l = \rho, \quad (b)\ \ \partial^m B_{mn} = 0. \tag{2.2}$$

The strengths $E_k,\ B_{mn}$ have conventional potential representation:

$$E_l = -\partial_t A_l - \partial_l \phi, \quad B_{mn} = \frac{1}{2}\epsilon_{mnkl}(\partial_k A_l - \partial_l A_k),\ \ l,m,n = 1,2,3,4. \tag{2.3}$$

Within potential representation (2.3) Eq.(2.1b) are transformed into identity. Under the condition of Lorentz-gauge equations $\partial_t \phi + \partial^k A_k = 0$, Eq.(2.1a) is reduced into the wave equations for the potentials

$$(\partial_t^2 - \partial_r^2)\phi = \rho, \quad (\partial_t^2 - \partial_r^2)A_l = j_l,\ \ l = 1,2,3,4. \tag{2.4}$$

Define the vector of momentum density, analogue of the Pointing vector, and the energy density

$$\Pi_k = \epsilon_{klmn} E_l B_{mn}, \quad \Pi_0 = \frac{1}{2}(E^2 + B^2),\ \ E^2 = E^k E_k,\ \ B^2 = \frac{1}{2} B^{mn} B_{mn}. \tag{2.5}$$

They obey the continuity equation

$$\partial_t \Pi_0 + \partial_k \Pi_k = 0. \tag{2.6}$$

One may notice, that the Maxwell equations formulated above are nothing else than the Maxwell equations in five dimensional space-time [7].

# 3 Mechanics with Cubic Characteristic Polynomial

**3.1 Evolution equations for inner momenta.**

In the previous section we have seen that the one dimensional relativistic equations of motion can be decomposed into two Newtonian equations with different evolution parameters. Inversely, two Newtonian equations can be coupled into one relativistic equation. In this section we generalize this scheme to the case of three coupled equations. Consider set of three Newtonian equations written with respect to different evolution parameters

$$(a)\ \frac{dp}{dt_p} = -\frac{dV(r)}{dr},\ \frac{dr}{dt_p} = \frac{p}{m};\ \ (b)\ \frac{dq}{dl_q} = -\frac{dV(r)}{dr},\ \frac{dr}{dl_q} = \frac{q}{\mu};\ \ (c)\ \frac{dh}{dl_h} = -\frac{dV(r)}{dr},\ \frac{dr}{dl_p} = \frac{h}{\nu}, \tag{3.1}$$

where the parameters of evolution $l_q, l_h$ are taken in unit of distance. The momenta $q,\ h$ and the mass-parameters $\mu,\ \nu$ have units of the energy. The momentum $p$ has to be proportional to the physical momentum of the particle $P$, which is proportional to the velocity. Hence $p$ has to be equal to zero at the rest. The momentum $p$ closely related with the kinetic energy of the particle whereas the momenta $q,\ h$ are related with the internal energy. At the rest $p = 0$ and $q(p = 0) = \mu,\ h(p = 0) = \nu$.

Now we are seeking the set of three equations written with respect to unique parameter of evolution. The desired system of equations is given by

$$(a)\ \frac{dp}{ds} = -\frac{dV(r)}{dr}\frac{q}{\mu}\frac{h}{\nu};\ \ (b)\ \frac{dq}{ds} = -\frac{dV(r)}{dr}\frac{p}{m}\frac{h}{\nu};\ \ (c)\ \frac{dh}{ds} = -\frac{dV(r)}{dr}\frac{p}{m}\frac{q}{\mu};\ \ (d)\ \frac{dr}{ds} = \frac{p}{m}\frac{q}{\mu}\frac{h}{\nu}. \tag{3.2}$$

In the case of stationary potential field Eqs.(3.2) imply three constants of motion, triplet of energies:

$$\mathcal{E}_p = \frac{p^2}{2m} + V(r), \quad \mathcal{E}_q = \frac{q^2}{2\mu} + V(r), \quad \mathcal{E}_h = \frac{h^2}{2\nu} + V(r)$$

By replacing the parameter of evolution $s$ by $t_p, l_q, l_h$ according to the rules

$$ds\frac{qh}{\mu\nu} = dt_p, \quad ds\frac{ph}{m\nu} = dl_q \quad \text{and} \quad ds\frac{pq}{m\mu} = dl_h.$$

we come from Eq.(3.2a,b,c) into Eq.(3.1a,b,c). There exist a correspondence between equations (3.2a,b,c) and equations (1.8). In fact, by replacing the parameter of evolution in (3.2a,b) by $ds\frac{h}{\nu} = d\tau$, we come to Eqs.(1.8). Notice, equations (3.2a,b,c) formally can be transformed into differential equations for Jacobi elliptic functions:

$$(a)\ \frac{d}{d\varphi}\frac{p}{\sqrt{m\mu}} = \frac{q}{\mu}\frac{h}{\nu}, \quad (b)\ \frac{d}{d\varphi}\frac{q}{\mu} = \frac{p}{\sqrt{m\mu}}\frac{h}{\nu}, \quad (c)\ \frac{d}{d\varphi}\frac{h}{\nu} = \frac{\mu}{\nu}\frac{p}{m}\frac{q}{\mu}, \quad \frac{d\varphi}{ds} = \frac{1}{\sqrt{m\mu}}\frac{dV}{dr}.$$

Then the solutions are expressed via Jacobi elliptic functions

$$sc(\varphi, k) = \frac{p}{\sqrt{m\mu}}, \ nc(\varphi, k) = \frac{q}{\mu}, \ dc(\varphi, k) = \frac{h}{\nu}, \ k^2 = \frac{\mu}{\nu}.$$

**3.2 Evolution equations for outer momenta.**

Now, we shall explore the following task:
define the mapping from the set of inner-momenta onto the set of outer-momenta. Evidently, in quality of the inner-momenta we consider the set of variables $p, q, h$. The main problem is now to define *the set of outer-momenta* [6].

The momentum by its definition has to be proportional to the velocity (3.2d). Following this principle we obtain the first member of the desired set of outer momenta:

$$P = m\frac{dr}{ds} = p\frac{q}{\mu}\frac{h}{\nu}. \tag{3.3}$$

For the next calculations we shall use the variables

$$\mathrm{p}^2 = \frac{p^2}{2m}, \quad \mathrm{q}^2 = \frac{q^2}{2\mu}, \quad \mathrm{h}^2 = \frac{h^2}{2\nu}.$$

Define triple product of the squares

$$\mathcal{P} = \mathrm{p}^2\mathrm{q}^2\mathrm{h}^2 = P^2\frac{\mu\nu}{8m} = P^2\frac{\nu c^2}{2}, \quad \text{because} \quad \mu = 4mc^2. \tag{3.4}$$

Evolution equations for the squares of the quantities $\mathrm{p}, \mathrm{q}, \mathrm{h}$ are

$$\frac{d}{ds}\mathrm{p}^2 = \frac{d}{ds}\mathrm{q}^2 = \frac{d}{ds}\mathrm{h}^2 = \frac{e}{m}(P^k E_k). \tag{3.5}$$

For the sake of convenience let us consider the evolution with respect to parameter $\psi$ defined by

$$d\psi = eE\sqrt{\frac{8}{m\mu\nu}}ds = E\frac{e}{mc}\sqrt{\frac{2}{\nu}}ds,$$

where $E$ is projection of the vector of electric field on the direction of motion. Re-write Eqs.(3.5) as follows

$$\frac{d\mathrm{p}^2}{d\psi} = \frac{d\mathrm{q}^2}{d\psi} = \frac{d\mathrm{h}^2}{d\psi} = \mathcal{P}. \tag{3.6}$$

By using these equations calculate the derivative of $\mathcal{P}^2$ with respect to $\psi$. The result label by $2\overset{2}{\mathcal{P}}_0$ by introducing a new quantity

$$\overset{2}{\mathcal{P}}_0 = \frac{1}{2}(\mathrm{p}^2\mathrm{q}^2 + \mathrm{q}^2\mathrm{h}^2 + \mathrm{h}^2\mathrm{p}^2). \tag{3.7}$$

Further, evaluate the derivative of $\overset{2}{\mathcal{P}}_0$. The result is given by $\mathcal{P}\ \overset{1}{\mathcal{P}}_0$, where

$$\overset{1}{\mathcal{P}}_0 = \frac{1}{3}(\mathrm{p}^2 + \mathrm{q}^2 + \mathrm{h}^2). \tag{3.8}$$

The next differentiation of $\overset{1}{\mathcal{P}}_0$ brings up the system of equations for the set of variables $\{\mathcal{P}, \overset{1}{\mathcal{P}}_0, \overset{2}{\mathcal{P}}_0\}$:

$$\frac{d\mathcal{P}}{d\psi} = \ \overset{2}{\mathcal{P}}_0, \quad \frac{d\overset{2}{\mathcal{P}}_0}{d\psi} = 3\ \mathcal{P}\ \overset{1}{\mathcal{P}}_0, \quad \frac{d\overset{1}{\mathcal{P}}_0}{d\psi} = \ \mathcal{P}. \tag{3.9}$$

The set of formulae (3.4), (3.7), (3.8) form the desired mapping from the triplet of variables $\{\mathrm{p}^2,\ \mathrm{q}^2,\ \mathrm{h}^2\}$ onto the triplet of variables $\{\mathcal{P}, \overset{1}{\mathcal{P}}_0, \overset{2}{\mathcal{P}}_0\}$. Notice this mapping is nothing else than the set of Vieta's formulae for the cubic polynomial

$$X^3 - 3\overset{1}{\mathcal{P}}_0\ X^2 + 2\overset{2}{\mathcal{P}}_0\ X - \mathcal{P}^2 = 0. \tag{3.10}$$

The system of equations (3.9) admits two first integrals of motion

$$R_1 = -2\overset{2}{\mathcal{P}}_0 + 3(\overset{1}{\mathcal{P}}_0)^2, \quad R_0 = (\overset{1}{\mathcal{P}}_0)^3 - R_1\ \overset{1}{\mathcal{P}}_0 - \mathcal{P}^2. \tag{3.11}$$

By using the constants of motion one may present solution of Eqs.(3.9) as an integral. From (3.9) and (3.11) it follows

$$\frac{d\overset{1}{\mathcal{P}}_0}{d\psi} = \sqrt{4(\overset{1}{\mathcal{P}}_0)^3 - 4R_1\overset{1}{\mathcal{P}}_0 - 4R_0}, \quad \psi = \int \frac{dy}{\sqrt{4y^3 - 4R_1 y - 4R_0}}.$$

Consequently, the solution $\overset{1}{\mathcal{P}}_0 = \rho(\psi; 4R_1, 4R_0)$ is expressed via Weierstrass elliptic function [8].

In conclusion of this section let us re-write Eqs.(3.9) with respect to the parameter of evolution $s$:

$$\frac{dP_k}{ds} = \frac{e}{mc^2}\frac{2}{\nu}E_k\ \overset{2}{\mathcal{P}}_0, \quad \frac{d\overset{2}{\mathcal{P}}_0}{ds} = 3\ \frac{e}{m}(E^k P_k)\ \overset{1}{\mathcal{P}}_0, \quad \frac{d\overset{1}{\mathcal{P}}_0}{ds} = \ \frac{e}{m}(E^k P_k). \tag{3.12}$$

# 4 Tensorial form of dynamic equations for triple of inner and outer momenta

### 4.1 Properties of the ternary mapping in $4D$ Euclidean space

In the previous section we established the mapping only between the lengths of the momenta. Now we shall extend this scheme within the framework of tensorial calculus. Noteworthy, the $4D$ Euclidean space possesses an advantage to build ternary mapping from three vectors onto one vector. In this section we shall use this relevant feature of the $4D$ Euclidean space. Firstly, consider the following mapping

$$P^k = \epsilon^{klmn} p_l q_m h_n, \quad k, l, m = 1, 2, 3, 4.$$

This composition law yet does not satisfy to our purposes because the square of the resulting vector cannot be presented as a product of the squares of the three vectors. For that purpose we extend this composition law as follows

$$\mathcal{P}^k = \epsilon^{klmn}\mathrm{p}_l\mathrm{q}_m\mathrm{h}_n + \mathrm{p}_k(\mathrm{q}\cdot\mathrm{h}) + \mathrm{q}_k(\mathrm{h}\cdot\mathrm{p}) - \mathrm{h}_k(\mathrm{p}\cdot\mathrm{q}) \tag{4.1}$$

Then, by direct calculations it can be proved the following **Theorem** holds true:

*The squared length of the vector $\mathcal{P}^k$ is equal to ternary product of the squared length of the vectors* $\mathrm{p}_l, \mathrm{q}_m, \mathrm{h}_n$, *i.e.,*

$$\mathcal{P}^2 = \mathrm{p}^2\mathrm{q}^2\mathrm{h}^2. \tag{4.2}$$

**4.2 Evolution equations for internal and external momenta inside magnetic fields.**

The dynamic equations for inner- and outer-momenta formulated in Section 3 have been stuck to the direction of motion. The purpose of this section is to release these equations of motion from the elected direction. The main results of this section are given by two systems of equations, first, for the inner momenta, and the second, for the outer momenta. The equations for the inner- momenta are given by the following system of equations

$$\frac{d}{ds}\mathrm{p}^l = \frac{e}{2mc}\Big( \sqrt{\frac{2}{\nu}}\big( \epsilon^{albc}E_a\mathrm{q}_b\mathrm{h}_c + E_l(\mathrm{h}\cdot\mathrm{q}) + \mathrm{h}_l(E\cdot\mathrm{q}) - \mathrm{q}_l(E\cdot\mathrm{h}) \big) + \epsilon^{klmn}\mathrm{p}_l B_{mn} \Big) \tag{4.3a}$$

$$\frac{d}{ds}\mathrm{q}^m = \frac{e}{2mc}\Big( \sqrt{\frac{2}{\nu}}\big( \epsilon^{abmc}E_a\mathrm{p}_b\mathrm{h}_c + \mathrm{h}_m(E\cdot\mathrm{p}) + E_m(\mathrm{p}\cdot\mathrm{h}) - \mathrm{p}_m(E\cdot\mathrm{h}) \big) + \epsilon^{klmn}\mathrm{q}_l B_{mn} \Big) \tag{4.3b}$$

$$\frac{d}{ds}\mathrm{h}^n = \frac{e}{2mc}\Big( \sqrt{\frac{2}{\nu}}\big( \epsilon^{abcn}E_a\mathrm{p}_b\mathrm{q}_c + \mathrm{q}_n(E\cdot\mathrm{p}) + \mathrm{p}_n(E\cdot\mathrm{q}) - E_n(\mathrm{q}\cdot\mathrm{p}) \big) + \epsilon^{klmn}\mathrm{h}_l B_{mn} \Big) \tag{4.3c}$$

From this system the following system of equations for the outer- momenta is derived

$$\frac{d}{ds}P^k = \frac{e}{m}E^k \overset{2}{\mathcal{P}}_0 \frac{2}{\nu c^2} + \frac{e}{2m}\epsilon^{klmn}P_l B_{mn}, \; k = 1,2,3,4. \tag{4.4}$$

The magnetic field does not produce a work therefore this force is given by orthogonal to the momentum expression. The structure of the ponder-motive force produced by magnetic field $B$ in the equations are defined via operation of the vector product which is defined by using the $\epsilon$-tensor. The analogue of the formula $[\vec{p}\times\vec{H}]^k = \epsilon^{klm}p_l H_m, \quad k,l,m = 1,2,3;$ in $4D$-space is $\epsilon^{klmn}p_l B_{mn}, \quad k,l,m = 1,2,3,4.$

# 5 Analogues of Hamilton-Jacobi and Klein-Gordon equations

Now let explore a important task on existence of Hamilton-Jacobi formulation of the dynamic equations (4.4. The following **Theorem** holds true:

*Let $s = s(t, r_1, r_2, r_3, r_4)$, and suppose that the momentum does not depend explicitly of $s$, so that,*

$$\frac{d}{ds}P_k = \frac{dt}{ds}\frac{\partial}{\partial t}P_k + \frac{dr_l}{ds}\frac{\partial}{\partial r_l}P_k, \; k = 1,2,3,4. \tag{5.1}$$

*Then within the definitions*

$$\overset{1}{\mathcal{P}}_0 = -\partial_t S - e\phi, \quad P_k = \partial_k S - eA_k \tag{5.2}$$

*Eqs.(4.6) are transformed into an identity.*

Now by substituting formulae

$$\overset{1}{\mathcal{P}}_0 = -\partial_t S - e\phi, \quad P_k = \partial_k S - eA_k$$

into the cubic polynomial equation (3.11), and remembering (3.4),we come to the Hamilton-Jacobi equation for the function $S$:

$$(-\frac{\partial S}{\partial t} - e\phi)^3 - R_1(-\frac{\partial S}{\partial t} - e\phi) - \frac{\nu c^2}{2}(\frac{\partial S}{\partial r_k} - eA^k)(\frac{\partial S}{\partial r^k} - eA_k) = R_0. \tag{5.10}$$

**Analogue of Klein-Gordon equation.**

As soon as we found the Hamilton-Jacobi equation, by using conventional correspondence formulae between Hamilton-Jacobi and Klein-Gordon equations we are able to postulate an analogue of the Klein-Gordon equation:

$$( \ (i\hbar\frac{\partial}{\partial t}-e\phi)^3-R_1(i\hbar\frac{\partial}{\partial t}-e\phi)-\frac{\nu c^2}{2}(-i\hbar\frac{\partial}{\partial r_k}-eA^k)(-i\hbar\frac{\partial}{\partial r^k}-eA_k)\ )\Psi=R_0\ \Psi. \quad (5.11)$$

# 6 Electromagnetic fields with triple strengths

Formula (2.5) for the energy density $\Pi_0$ is quite similar to the formula for $P_0$ in (1.7). The difference is that in (2.5) one deals with the density of the energy. In Section 3 we extended the formula (1.7) to the three terms (formula (3.8)). This analogue leads us to the idea *to seek field equations for triple fields, the energy density of which are defined as the sum of the energies of the three fields:* $E_l, B_m, K_n,\ l,m,n=1,2,3,4.$

We postulate the following analogue of the Maxwell equations for the triple strengths:

$$\frac{\partial}{\partial t}\ E^l=\frac{1}{2}\epsilon^{klmn}\partial_k(\mathcal{B}_m\mathcal{K}_n),\quad \frac{\partial}{\partial t}\ \mathcal{B}^m=\frac{1}{2}\epsilon^{klmn}\partial_k(E_l\mathcal{K}_n),\quad \frac{\partial}{\partial t}\ \mathcal{K}^n=\lambda^2\frac{1}{2}\epsilon^{klmn}\partial_k(E_l\mathcal{B}_m). \quad (6.1)$$

The following **Proposition** holds true:

*Let the field equations for* $E,\mathcal{B},\mathcal{K}$ *are given by Eqs.(6.1) where the energy and the momentum densities are defined by*

$$\Pi_0=\frac{1}{2}(\ E^2+\mathcal{B}^2+\frac{1}{\lambda^2}\mathcal{K}^2),\quad \Pi^k=\epsilon^{klmn}E_l\mathcal{B}_m\mathcal{K}_n. \quad (6.2)$$

*Then the quantities* $\Pi_0,\ \Pi^k,\ k=1,2,3,4$ *obey the following continuity equation*

$$\frac{\partial}{\partial t}\Pi_0=(\nabla_k\Pi^k). \quad (6.3)$$

In order to obtain some correspondence with the Maxwell equations (2.1) tend $\lambda^2$ to zero. Then $\mathcal{K}_n$ does not depend of $t$ explicitly. Furthermore, suppose $\mathcal{K}_n$ is a gradient of some scalar function $\Phi(x_1,x_2,x_3,x_4)$, so that,

$$\mathcal{K}_n=\partial_n\Phi,\quad \text{with,}\quad (\partial^l\Phi)(\partial_l\Phi)=4. \quad (6.4)$$

Then Eqs.(6.1a,b) are reduced as follows:

$$(a)\ \frac{\partial}{\partial t}\ E^k=\frac{1}{2}\epsilon^{klmn}\partial_l B_{mn}+j^k,\quad (b)\ \frac{\partial}{\partial t}\ \mathcal{B}^m=\frac{1}{2}\epsilon^{klmn}\mathcal{K}_l\nabla_k(E_l), \quad (6.5)$$

where

$$B_{mn}=\frac{1}{2}(\mathcal{K}_m\mathcal{B}_n-\mathcal{B}_m\mathcal{K}_n). \quad (6.6)$$

From the second equation of Eqs.(6.5) it follows

$$\frac{\partial}{\partial t}\ (\mathcal{B}^m\mathcal{K}_m)=0. \quad (6.7)$$

i.e, $\mathcal{B}_m\mathcal{K}^m$ does not depend of time. From (6.6) and (6.7) we get

$$2\partial_t\ \mathcal{B}_m=-\partial_t\ (B_{mn}\mathcal{K}^n). \quad (6.8)$$

Combination (6.5b) with (6.8) gives

$$\mathcal{K}_n\frac{\partial}{\partial t}\ B^{mn}=-\epsilon^{mnkl}\mathcal{K}_n\partial_k E_l. \quad (6.9)$$

If in the equation (6.4) we do not fix boundary conditions, then $K_n$ remains arbitrary, consequently Eq.(6.9) is reduced to Eq.(2.1b).

**Conclusions**

We presented a new generalization of the relativistic dynamics of the charged particle. The covariant equations of motion are formulated in $4D$-space which possesses an advantage to use quaternion or tensorial analysis. The new dynamic equations admit Hamilton-Jacobi formulation which is important in order to build a quantum analogue of the dynamics. An essential ingredient of the theory is field equations for new kind of electromagnetic fields with triple strengths.

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