# Generalized Fundamental Polynomials

P.Kh. Atanasova

e-mail: poli@jinr.ru, Laboratory of Information Technologies, JINR, Dubna

## 1. Interpolation with algebraic polynomials

Using the fundamental theorem of algebra it is easy to show that for every $A$-polynomial of degree $n$, the identity

$$A(x) \equiv \sum_{k=0}^{n} A(x_k) l_k(x) \tag{1}$$

holds true.

## 2. Interpolation with trigonometric polynomials

For every $T$-polynomial of order $n$ the identity

$$T(x) \equiv \frac{1}{2n+1} \sum_{k=0}^{2n} T(x_k) \frac{\sin \frac{2n+1}{2}(x - x_k)}{\sin \frac{x-x_k}{2}} \tag{2}$$

holds true.

## 3. Interpolation with exponential polynomials

Similarly to 2. we could make a presentation for every exponential polynomial ($E$-polynomial) $E_n(x)$ on the basis $(1, shx, chx, sh2x, ch2x, \ldots, shnx, chnx)$ or $(1, e^x, e^{-x}, e^{2x}, e^{-2x}, \ldots, e^{nx}, e^{-nx})$ in the form

$$E(x) \equiv \sum_{k=0}^{2n} E(x_k) h_k(x)\,, leqno(3)$$

where the fundamental $E$-polynomials $h_k(x)$ can be written as follows:

$$h_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{2n} \frac{sh \frac{x-x_i}{2}}{sh \frac{x_k - x_i}{2}}\,, \quad h_k(x_i) = \delta_{ki}\,.$$

## 4. The most general interpolation problem

Let $X$ be a linear space of functions and $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x) \in X$. Let also $L_0, L_1, \ldots, L_n$ be linear functionals defined in $X$. It is well known that the necessary and sufficient condition for the general interpolation problem

$$L_k(G_n) = L_k(f), \quad k = \overline{0, n}\,, \tag{4}$$

is to have a unique solution as a generalized polynomial ($G$-polynomial) $G_n(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \ldots + a_n\varphi_n(x)$ on the basis $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$ for every $f(x) \in X$ is

$$\Delta = det[L_k(\varphi_i)] \neq 0\,. \tag{5}$$

If we chose the functionals $L_k$ as $L_k(g) = g(x_k)$, $k = \overline{0, n}$ when $x_0, x_1, \ldots, x_n$ are different points in interval $[a, b]$, then the condition (5) shows that the basic functions $\{\varphi_k(x)\}_{k=0}^n$ form a Chebishev system for $[a, b]$. Many special interesting cases of choice of $L_0, L_1, \ldots, L_n$ are considered.

***Lemma:*** *If $\bar{G}(x)$ and $\bar{\bar{G}}(x)$ are $G$-polynomials on the basis $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$ and $L_k(\bar{G}) = L_k(\bar{\bar{G}})$, $k = \overline{0, n}$ then $\bar{G}(x) \equiv \bar{\bar{G}}(x)$.*

The solution of the most general interpolation problem (4) can be written in the form

$$G_n(x) = \sum_{k=0}^{n} L_k(f) \Phi_k(x)\,, leqno(6)$$

where $\Phi_k(x)$, $k = \overline{0, n}$, are fundamental generalized $G$-polynomials on the basis $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$, for which the equalities

$$L_i(\Phi_k) = \delta_{ik}$$

hold true. This fact follows from the presentation (6) of $G_n(x)$ in the determinant form.

$$\Phi_k(x) = \frac{1}{det[L_k(\varphi_i)]} \begin{vmatrix} L_0(\varphi_0) & \ldots & L_0(\varphi_n) \\ \ldots & \ldots & \ldots \\ L_{k-1}(\varphi_0) & \ldots & L_{k-1}(\varphi_n) \\ \varphi_0(x) & \ldots & \varphi_n(x) \\ L_{k+1}(\varphi_0) & \ldots & L_{k+1}(\varphi_n) \\ \ldots & \ldots & \ldots \\ L_n(\varphi_0) & \ldots & L_n(\varphi_n) \end{vmatrix}$$

The main result is the following.

**For every $G$-polynomial the identity**

$$G(x) \equiv \sum_{k=0}^{n} L_k(G) \Phi_k(x) \tag{7}$$

**holds true.**

The conclusion of the theorem follows from the above lemma and (6).

This result (7) generalizes the formulas (1), (2), (3) for algebraic, trigonometric and exponential polynomials.

Published in Sci. works of the Univ. of Plovdiv, Vol 35, 2008.