On Multidimensional Solitons in Gauge-invariant Nonlinear Sigma Models

I.L. Bogolubsky, A.A. Bogolubskaya

e-mail: bogolubs@jinr.ru, abogol@jinr.ru. Laboratory of Information Technologies, JINR, Dubna

Motivation. Nonlinear sigma models (NSMs) are of great importance in the modern mathematical physics due to their universality: they appear in various branches of fundamental science. Classical NSMs describe evolution in time of N-component unit isovector field $s_a(\mathbf{x},t)$ in $(D+1)$-dimensional space-time $(a=1,...,N+1)$; field manifolds of these models are unit spheres S^N. The most interesting cases correspond to $D=2,3$ and $N=2,3$.

Below we discuss the $A3M$ model with $N=2$ and $D=2$, introduced in [1], and the $A4YM$ model with $N=3$ and $D=3$, introduced in [2]. The $A4YM$ model is the straightforward extension of the $A3M$ model. On the other hand, one can see deep resemblance of the $A4YM$ model with the bosonic sector of the reduced electroweak Salam-Weinberg theory, widely known as SU2-Higgs model, in which radial degree of freedom of the Higgs field is frozen (see the A4YM Lagrangian below). In fact, our gauged NSMs include: i) unit length scalar $(N+1)$-component field, with values on S^N :

$$s_1{}^2+...s_{N+1}{}^2=1 \quad (N=2,3),$$

interacting with ii) vector field with $U(1)$ or $SU(2)$ symmetry (Maxwell or Yang-Mills).

A3M model in 2 dimensions. Consider minimal interaction of the S^2 scalar field (A3-field) with the Maxwell field $A_\mu(x)$. The resulting "A3M model" is described by the gauge-invariant Lagrangian:

$$\begin{gathered}\mathcal{L}=\eta^2\left(\bar{\mathcal{D}}_\mu s_-\mathcal{D}^\mu s_+ +\partial_\mu s_3\partial^\mu s_3\right)-V(s_a)-\frac{1}{4}F^2_{\mu\nu},\\ \bar{\mathcal{D}}_\mu=\partial_\mu+igA_\mu,\quad \mathcal{D}_\mu=\partial_\mu-igA_\mu,\\ s_+=s_1+is_2,\quad s_-=s_1-is_2,\\ F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\\ V(s_a)=\beta(1-s_3^2),\end{gathered}\tag{1}$$

where β^2,η^2 are constants, $[\eta^2]=L^{(1-D)}$, $[\beta^2]=L^{-(1+D)}$, g is a coupling constant, $[g^2]=L^{(D-3)}$, $\mu,\nu=0,1,...,D$, and summation over repeated indices μ,ν is meant. The A3M model is a gauge-invariant extension of the classical Heisenberg antiferromagnet model with the *"easy-axis"* anisotropy; it possesses global $Z(2)$ and local $U(1)$ symmetries which might be connected with some surprising remarkable properties of its localized solutions.

The Euler-Lagrange equations of the model in dimensionless form are obtained by rescaling $x_\mu\to g^{-1}\eta^{-1}x_\mu$, $A_\mu\to\eta^{-1}A_\mu$ (we denote $p=\beta^2g^{-2}\eta^{-4}$) and take the simplest form if the Lorentz gauge, $\partial_\mu A^\mu=0$,

$$\begin{gathered}\partial_\mu\partial^\mu s_i+[\partial_\mu s_a\partial^\mu s_a+2A_\mu j^\mu+p(s_3^2-\delta_{i3})+\\ A_\mu A^\mu(s_1^2+s_2^2-\delta_{1i}-\delta_{2i})]s_i-\\ -2A_\mu(\delta_{2i}\partial^\mu s_1-\delta_{1i}\partial^\mu s_2)=0,\\ j_\mu=s_2\partial_\mu s_1-s_1\partial_\mu s_2,\\ \partial_\mu\partial^\mu A_\nu+2j_\nu+2(s_1^2+s_2^2)A_\nu=0,\end{gathered}\tag{2}$$

where $\mu,\nu=0,1,...,D,\quad i=1,2,3.$

In angular variables, $s_1=\sin\theta\cos\phi$, $s_2=\sin\theta\sin\phi$, $s_3=\cos\theta$, the Lagrangian density reads:

$$\begin{gathered}g^{-2}\eta^{-4}\mathcal{L}=\partial_\mu\theta\partial^\mu\theta+\sin^2\theta\ [\ \partial_\mu\phi\partial^\mu\phi-\\ -2A_\mu\partial^\mu\phi+A_\mu A^\mu-p\ ]-\frac{1}{4}F^2_{\mu\nu}\end{gathered}\tag{3}$$

and the Euler-Lagrange equations become:

$$\begin{gathered}\partial_\mu\partial^\mu\theta+\frac{1}{2}\sin 2\theta\ [\ p-\partial_\mu\phi\partial^\mu\phi+2A_\mu\partial^\mu\phi-\\ -A_\mu A^\mu\ ]=0,\\ \partial_\mu\left[\sin^2\theta(\partial^\mu\phi-A^\mu)\right]=0,\\ \partial_\mu\partial^\mu A_\nu+2j_\nu+2A_\nu\sin^2\theta=0,\\ j_\nu=-\sin^2\theta\partial_\nu\phi.\end{gathered}\tag{4}$$

Time-independent soliton solutions $\phi(x)=\phi(\mathbf{x})$, $A_0=0$, $A_k(x)=A_k(\mathbf{x})$, $\theta(x)=\theta(\mathbf{x}), k=1,...,D$, obey equations:

$$\begin{gathered}\partial_k^2\theta-\frac{1}{2}\sin 2\theta\left[p+(\partial_k\phi-A_k)^2\right]=0,\\ \partial_k\left[\sin^2\theta(\partial_k\phi-A_k)\right]=0,\\ \partial_k^2A_m+2\sin^2\theta(\partial_m\phi-A_m)=0,\end{gathered}\tag{5}$$

$(k,m=1,...,D$, summation over repeated k is meant). The localized distributions of unit isovector $s_a(\mathbf{x})$ in this model are divided into classes with different topological inidices ("charges") Q_t; solitons with nonzero topological charges are referred to as "topological solitons" [1]. We look for the topological solitons of the A3M model using the "hedgehog-like" ansatz for the A3-field

$$\begin{gathered}s_1=\cos m\chi\sin\theta(R),\quad s_2=\sin m\chi\sin\theta(R),\\ s_3=\cos\theta(R),\\ \sin\chi=\frac{y}{R},\quad\cos\chi=\frac{x}{R},\quad R^2=x^2+y^2,\end{gathered}\tag{6}$$

where m is an integer number. We use also the standard "vortex" ansatz for the vector field A_μ, describing localized distributions of a stationary magnetic field:

$$A_0 = 0,$$

$$A_1 = A_x = -ma(R)\frac{y}{R^2}, \quad A_2 = A_y = ma(R)\frac{x}{R^2}. \quad (7)$$

For them $\quad Q_t = m$.

After rescaling ($a = \alpha e^{-1}, \quad R = re^{-1}$), we calculate $\delta H/\delta\theta$ and $\delta H/\delta\alpha$, arriving at coupled equations for $\theta(r)$ and $\alpha(r)$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{m^2(\alpha-1)^2}{r^2} + p\right] = 0, \quad (8)$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} + 2\sin^2\theta(1-\alpha) = 0, \quad (9)$$

to be solved under the following boundary conditions:

$$\theta(0) = \pi, \quad \theta(\infty) = 0, \quad (10)$$

$$\alpha(0) = 0, \quad \frac{d\alpha}{dr}(\infty) = 0. \quad (11)$$

Using series expansion of $\theta(r)$ and $\alpha(r)$ at $r \to 0$, we find from Eqs. (8) and (9) for $m = 1$

$$\theta(r) = \pi - C_1 r + o(r),$$

$$\alpha(r) = r^2(E_1^2 - \frac{1}{4}C_1^2 r^2) + o(r^4),$$

and for $m = 2$

$$\theta(r) = \pi - C_2 r^2 + o(r^2),$$

$$\alpha(r) = r^2(E_2^2 - \frac{1}{12}C_2^2 r^4) + o(r^6).$$

We studied the problem (8)-(11) by various numerical methods, among them shooting technique, stabilization method. The method based on power and asymptotic series and on the analytic continuation technique (re-expansions and Pade approximants) was used as well [3].

Detailed numerical investigation shows [4] that solutions exist and are stable for the values of dimensionless anisotropy parameter

$$0 < p < p_{cr} \approx \; 0.41.$$

The plots of radial functions $\alpha(r)$ and $\theta(r)$ and corresponding distributions of energy density and magnetic field have been presented in [1]. Note that the asymptotic value $\alpha_\infty = \alpha(\infty)$ decreases monotonically as p is increased, with $\alpha_\infty \to 1$ when $p \to 0$. With the function $\alpha_\infty(p)$ at hand one can find the asymptotic form of the soliton solution for $r \to \infty$:

$$\theta(r) \approx \frac{T}{\sqrt{r}}\exp(-\sqrt{p}r), \quad T = const,$$

$$\alpha(r) \approx \alpha_\infty - (1-\alpha_\infty)\frac{T^2}{2rp}\exp(-2\sqrt{p}r).$$

The dependence of the soliton energy $E = 2\pi\int\mathcal{H}(r)rdr$ on p, where

$$\mathcal{H}(r) = \left(\frac{d\theta}{dr}\right)^2 + \sin^2\theta\left[p + \frac{m^2(\alpha-1)^2}{r^2}\right] + $$
$$+\frac{m^2}{2}\left(\frac{1}{r}\frac{d\alpha}{dr}\right)^2, \quad (12)$$

is depicted in Figure 1. It is important to note that $E(p) < 8\pi$ for $p < p_{cr} \approx 0.4088$ (recall that 8π is the energy value of the Belavin-Polyakov localized solutions in the 2D isotropic Heisenberg ferromagnet [5]). It means that for $0 < p < p_{cr}$ the string-like solutions of the A3M model describe spatially localized *bound* states of the $A3$- and the Maxwell fields, and hence it is natural to conjecture these $2D$ solitons to be stable for $p < p_{cr}$.

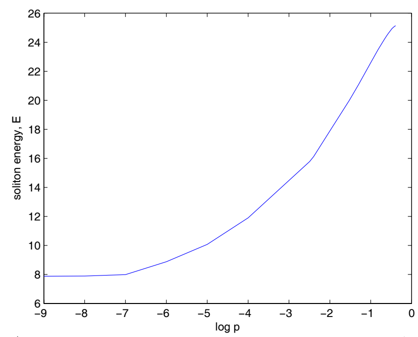

Figure 1: Soliton energy for various p, the $A3M$ model

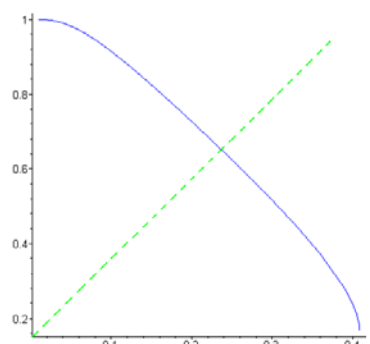

Figure 2: $\alpha(\infty)$ vs p for the $A3M$ model

It is interesting to note that the dependence $\alpha(\infty)$ on p proved to be surprisingly symmetric (see Figure 2). Presently the only way to explain such a symmetry is to refer to high $(U(1) \otimes Z(2))$ symmetry of the $A3M$ model (1).

Then we studied $Q_t = 2$ solitons. We have found that for all $\; 0 < p < p_{cr} \approx 0.41 \;$ their energies turned out to satisfy inequality

$$E_{sol}(Q_t = 2, p) < 2 * E_{sol}(Q_t = 1, p).$$

This means that two $Q_t = 1$ solitons attract to each other, forming the $Q_t = 2$ bound states as a result of initial configuration evolution.

A4YM model for D = 3. Further we shall consider another gauged sigma model, which describe minimal interaction of the easy-axis 4-component unit isovector field $q^\alpha(x^\mu)$ ("the A4-field") interacting with the vector $SU(2)$ Yang-Mills field $A^a_\mu(x^\nu)$.

The Lagrangian density of this ("the A4YM") model is:

$$\mathcal{L} = \mathcal{D}_\mu q^a \mathcal{D}^\mu q^a + \partial_\mu q^0 \partial^\mu q^0 - V(q^0) - \frac{1}{4}(F^a_{\mu\nu})^2,$$
$$\mathcal{D}_\mu q^a = \partial_\mu q^a + g\varepsilon^{abc} A^b_\mu q^c,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu,$$
$$V(q^0) = \beta[1 - (q^0)^2], \tag{13}$$

where $\alpha, \mu, \nu = 0, 1, 2, 3$; $a, b, c = 1, 2, 3$; β, g are coupling constants.

First we looked for stationary topological solitons of the A4YM model using the following ansatz for the A4- and the $SU(2)$ Yang-Mills fields:

$$q^0 = \cos\theta(R), \quad q^a = \sin\theta(R)\frac{x^a}{R},$$
$$R^2 = x^2 + y^2 + z^2, \quad A^a_0 = 0, \quad A^a_i = c(R)\varepsilon^{iak} x^k. \tag{14}$$

Then the Hamiltonian density distributions of localized field bunches are spherically symmetric:

$$\mathcal{H}_{st}(R) = \left(\frac{d\theta}{dR}\right)^2 + \frac{2\sin^2\theta}{R^2} + 4gc\sin^2\theta +$$
$$+2g^2c^2R^2\sin^2\theta + 6c^2 + \left(\frac{dc}{dR}\right)^2 R^2 +$$
$$+\frac{1}{2}g^2c^4R^4 + 4Rc\frac{dc}{dR} + 2gR^2c^3 + \beta\sin^2\theta. \tag{15}$$

Introduce dimensionless variables $r = gR$, $b(r) = g^{-1}cr^2$. Calculating $\delta\mathcal{H}/\delta\theta$ and $\delta\mathcal{H}/\delta r$, we get coupled equations $\quad (P = \frac{\beta}{g^2})$

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{2(b+1)^2}{r^2} + P\right] = 0,$$
$$\frac{d^2b}{dr^2} - \frac{2b}{r^2} - 2\sin^2\theta(1+b) - \frac{b^2}{r^2}(b+3) = 0. \tag{16}$$

When searching for localized solutions we set the following boundary conditions:

$$\theta(0) = \pi, \quad \theta(\infty) = 0,$$
$$b(0) = 0, \quad b(\infty) = B. \tag{17}$$

Solutions to above problem (16)-(17) would define localized distributions $q^\alpha(x^k)$ ($\alpha = 0, 1, 2, 3$, and $k = 1, 2, 3$) of the A4-field with unit topological charge, $Q_t = 1$. Here Q_t is the "mapping degree" of continuous maps $R^3_{comp} \to S^3$. However such solutions have not been found. Because of that we look for more general ansatz.

More general ansatz keeps the "hedgehog" form for q^α and a generalized expression for A_i:

$$A^a_i(x) = \epsilon_{aij} n_j C(R) R + (\delta_{ai} - n_a n_i)\frac{B(R)}{R} + n_a n_i \frac{E(R)}{R}. \tag{18}$$

However such ansatz should respect Lorentz gauge. Equating $\frac{\partial A^a_i}{\partial x_i} = 0$, we find $C(R) = B(R) + const$. Finally we obtain the ansatz

$$A^a_i(x) = \epsilon_{aij} n_j C(R) R + \delta_{ai}\frac{B(R)}{R} + \frac{G n_a n_i}{R}. \tag{19}$$

We calculate the Hamiltonian density $\mathcal{H}_{st}(R)$ for such ansatz using the computer algebra system Maple [6]. Equating variational derivatives $\frac{\delta\mathcal{H}_{st}(R)}{\delta C}$, $\frac{\delta\mathcal{H}_{st}(R)}{\delta B}$, $\frac{\delta\mathcal{H}_{st}(R)}{\delta\theta}$ to 0, we obtain coupled equations for radial functions $C(R), B(R), \theta(R)$. Their solutions (if exist) define localized soliton solutions to A4YM model, the study of coupled equations for $C(R), B(R), \theta(R)$ with unknown G is in progress.

Conclusions. In this paper we discussed the existence and properties of localized solutions of the $A3M$ model ($D = 2$) and the A4YM model ($D = 3$). Topological solitons of these models can be considered as soliton analogues of the so-called defect solutions: $2D$ strings-vortices in the Abelian Higgs model [7] and $3D$ 't Hooft-Polyakov "hedgehogs"-monopoles [8] correspondingly.

References

[1] I.L. Bogolubsky and A.A. Bogolubskaya. Phys. Lett.B, **395** (1997) 269-274.

[2] I.L. Bogolubsky., A.A. Bogolubskaya. Ann. Fond. Louis de Broglie, **23** (1998) 11-14.

[3] A.I. Bogolubsky and S.L. Skorokhodov. Programming and Computer Software, **30** (2004) 95-99.

[4] I.L. Bogolubsky, A.A. Bogolubskaya, A.I. Bogolubsky, S.L. Skorokhodov. In: Path Integrals - New Trends and Perspectives. Proceedings of the 9 International Conference /Janke W., Peltser A. (Eds.) Singapore: World Scientific, 2008.

[5] A.A. Belavin and A.M. Polyakov. Pis′ma v Zh.E.T.F., **22** (1975) 503-506 [JETP Lett., 22 (1975) 245-248].

[6] Maple 12 User Manual.(http://www.maplesoft.com/view.aspx?sid=5883).

[7] A. A. Abrikosov. Soviet Physics JETP, **5** (1957) 1174-1182; H.B. Nielsen and P. Olesen. Nuclear Physics, **B61** (1973) 45-61.

[8] G.'t Hooft. Nuclear Physics, **B79** (1974) 276-284; A.M. Polyakov. JETP Lett., **20** (1974) 194-195; A.M. Polyakov. JETP, **68** (1975) 1975-1990.