Effective Equation with Dispersion on the Correctness Boundary. Asymptotic Behaviour of Solution as $t \to \infty$

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We study the linear differential equations

$$u_{tt} = u_{xx} + u_{ttxx}, \qquad (1)$$

$$u_{tt} = u_{xx} + ibu_{xxx} + u_{ttxx}, \qquad (2)$$

$$u_{tt} = u_{xx} + ibu_{ttx} + u_{ttxx}, \qquad (3)$$

$$u_{tt} = u_{xx} - u_{xxxx}. \qquad (4)$$

The Cauchy problem with discontinuous initial conditions:

$$u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0, \end{cases} \qquad u_t(x,0) = 0, \qquad (5)$$

is solved. The problem arises in the study of wave motion in periodic stratified media [1,2].

It was proved [3], that equation (1) has an exotic (for the linear equation) asymptotics of a breather-type. In [4] deformation of the breather after adding a lower-order term with a complex coefficient was studied. For equation (3) with $b = 1$ the asymptotic behaviour of the solution at $t \to \infty$ was investigated. The proven asymptotic formulae confirmed the occurrence of deformation processes evidenced by numerical simulations. The starting point of the asymptotic study was the integral representation of the solution to the problem (3), (5):

$$u(x,t) = -\frac{1}{4\pi i} \int\limits_{\Gamma} \left(\exp(-\frac{ist}{\sqrt{1 - bs + s^2}} - ixs) + \exp(\frac{ist}{\sqrt{1 - bs + s^2}} - ixs) \right) \frac{ds}{s}.$$

The contour Γ goes along the real line except for a neighborhood of zero where the pole is rounded in the upper half plane over a semicircle of small radius.

The problem (3), (5) is well-posed in Petrovskii sense [5] when $|b| \leq 2$. In this work we report results of an additional study of the solution behaviour of the (3), (5) problem on the correctness boundary $b = 2$. In this case the integral representation changes to:

$$u(x,t) = -\frac{1}{4\pi i} \int\limits_{\Gamma} \left(\exp(-\frac{ist}{1 - s} - ixs) + \exp(\frac{ist}{1 - s} - ixs) \right) \frac{ds}{s} = I_1 + I_2.$$

The radicals in the integral representation of the solution vanished what simplifies the asymptotic proof problem. However, we have to prove everything again. Moreover, this time there is not any more rapid exponentially fast recovery of the limit values outside an interval $|x| < dt$, for finite d, which significantly complicates choosing the proper numerical algorithm.

In the case $b = 2$ we proved four theorems describing the asymptotics of the (3), (5) problem solution at various positive x values.

Theorem 1. *For* $b = 2, \quad t \to \infty, \quad |x| < ct^{-1}$

$$\Re(u(x,t)) = \frac{1}{2} + \frac{\text{sign}(x)}{2} \cos t + \mathrm{O}(xt).$$

By this the existence of breather-type solution in a small neighborhood of zero is stated. The asymptotics of the considered problem solution at $x > ct^{-1}$ and large enough t is determined by the integral I_2.

Theorem 2. *For* $b = 2, \quad t \to \infty, \quad x > ct^{-1}$, *the following relation holds*

$$I_1 = 1/2 + O(\exp(-\sqrt{xt}/2)).$$

The following theorem describes the behaviour of the considered problem solution near the characteristic.

Theorem 3. *For* $b = 2, \quad t \to \infty \quad |x - t| < c\sqrt{t}$

$$u(x,t) = \frac{3}{4} + \frac{e^{-i\pi/4}}{8\sqrt{\pi t}} + O(t^{-3/2}) + \frac{(x-t)}{4\sqrt{\pi t}} e^{i\pi/4}(1 + O(t^{-1})) + \frac{1}{8\sqrt{\pi t}} \exp(-i(x + t + 2\sqrt{xt} - \pi/4))(1 + O(\frac{x-t}{t})).$$

For the remaining $x > 0$ the next theorem holds.

Theorem 4. *For* $b = 2, \quad t \to \infty, x > ct^{-1}$, $|x - t| > c\sqrt{t}$

$$u(x,t) = d + \frac{ie^{-i(t+x)}(xt)^{1/4}(1 + O(1/\sqrt{xt}))}{2\sqrt{\pi}(x-t)} \times \left(cos(2\sqrt{xt} + \frac{\pi}{4}) + i\sqrt{\frac{t}{x}} sin(2\sqrt{xt} + \frac{\pi}{4}) \right).$$

Here $d = 1/2$ *iff* $x < t$ *and* $d = 1$ *iff* $x > t$.

In the numerical simulation the equation (3) was approximated by the second order accurate implicit difference scheme

$$\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\tau^2}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}+$$

$$bi\frac{u_{j+1}^{n+1}-2u_{j+1}^n+u_{j+1}^{n-1}-u_{j-1}^{n+1}+2u_{j-1}^n-u_{j-1}^{n-1}}{2h\tau^2}+$$

$$\frac{u_{j+1}^{n+1}-2u_{j+1}^n+u_{j+1}^{n-1}-2(u_j^{n+1}-2u_j^n+u_j^{n-1})}{h^2\tau^2}+$$

$$\frac{u_{j-1}^{n+1}-2u_{j-1}^n+u_{j-1}^{n-1}}{h^2\tau^2}.$$

The roots of the characteristic polynomial $\lambda^2-(2-c)\lambda+1$,

$$c=4\alpha^2(\sin\phi/2)^2h^2/(h^2+bh\sin\phi+4(\sin\phi/2)^2),$$

lie on the unit circle if $c\le 4$, which is equivalent to

$$(1-\alpha^2)h^2\sin^2(\phi/2)+(b\sin(\phi/2)+h\cos(\phi/2))^2+$$

$$(4-b^2)\sin^2(\phi/2)\ge 0.$$

This inequality holds for $\alpha \le 1$ and all $0<|b|\le 2$. In the computations performed we used $h=0.1, \alpha=\tau/h=1$. The difference scheme spectrum lies on the boundary of the unit disk. This prevents strong exponential instability. However, $\lambda_1(0)=\lambda_2(0)=1$, which leads to the L_2 instability $||G^n||=O(n)$, where G is the transition operator from one time level to another in the Cauchy problem. Nevertheless, power instability can not considerably distort the result on a bounded t-interval, $t\le T$. The computations are performed for $T=50$. Instead of the Cauchy problem, we solved an initial-boundary value problem on the x interval $[-100,0]$ with the following boundary conditions: $u(-100,t)=0$, $\Re(u(0,t))=1/2$. The value of $\Im(u(0,t))$ was determined from the difference approximation of equation (3) at $x=0$. In addition a symmetry property of the solution: $\Re(u(-x,t)=1-\Re(u(x,t))$, $\Im(u(-x,t)=\Im(u(x,t))$, was used. It is to be noted that the difference equation was rewritten as a system of two equations with real coefficients for the real and imaginary parts of u. The matrix stepwise pursuit was used at each time step. The computed pursuit matrix coefficients $D(j)=||d_{i,k}(j)||,\ 1\le i,k\le 2,\ j=1,...,1000$, suggest that the stepwise matrix pursuit is stable. For $b=2$

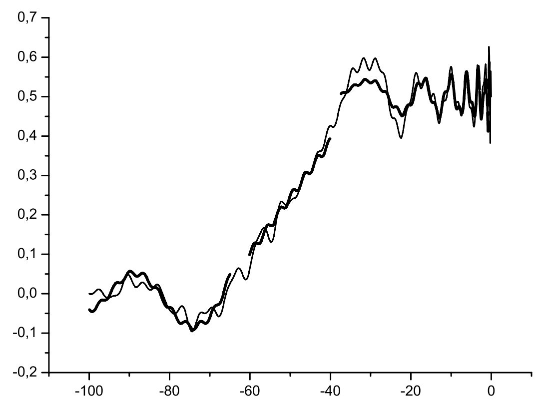
$$D_j=-\theta_jA,\ A=||a(i,j)||,\ a(1,1)=a(22)=1,$$

$$-a(1,2)=a(2,1)=h,$$

$$\theta_0=0,\ \theta_{j+1}=1/(2+h^2-(1+h^2)\theta_j).$$

The problem of the stepwise matrix pursuit stability is equivalent to the uniform boundedness in j elements of matrices $B_j=\theta_l\theta_{l-1}....\theta_jA^{l-j+1}$, $2\le j\le l$. In our case, $l=1000$, maximal in j value of $\max(|b_{1,1}(j)|+|b_{1,2}(j)|,|b_{2,1}(j)|+|b_{2,2}(j)|)$ is equal to 1.3605... The computations were performed by using the REDUCE system [9], well suited for the matrix manipulations, in particular for implementation of the stepwise matrix pursuit.

The asymptotic formulae are in good agreement with the results of the performed numerical experiments. Plots of the real part of the solution $u(x,50)$ for $b=2,-100\le x\le 0$ are given in the figure, where the thin line refers to the numerical solution and thick line to the asymptotics.



References

[1] N.S. Bakhvalov and M.E. Eglit, Dokl.Math. 61, 1-4 (2000)[ДАН.2000. Т.370. No.1. С.7-10].

[2] N.S. Bakhvalov and M.E. Eglit, Dokl.Math. 65,301-305 (2002)[ДАН. 2002. Т.383. No.6. С.742-746].

[3] S.I. Serdyukova, Dokl.Math. 67, 203-207 (2003) [ДАН. 2003. Т.389. No.3. С. 305-309].

[4] S.I. Serdyukova, Dokl.Math. 80, 1-7 (2009) [ДАН. 2009. Т.427. No.1. С. 17-23].

[5] I.G. Petrovskii Of the Cauchy Problem for Linear Partial Differential Equation Systems in Domain of Non-Analytical Functions. Büll. MSU, sektion A, vyp. 1 (1938).

[6] M.V. Fedoryuk, t Asymptotics: Integrals and Series(Nauka, Moscow, 1987)[in Russian].

[7] REDUCE User's Guide for Unix Systems. Version 3.8 by Winfried Neun, ZiB D-14195, Berlin-Dahlem, September 2004.