Solution of the Effective Equations of the Wave Propagation in Periodic Stratified Dispersive Media

S.I. Serdyukova
e-mail: sis@jinr.ru, Laboratory of Information Technologies, JINR, Dubna

For the first time the existence of exotic breather-type asymptotics is proved for solutions of non-standard linear differential equations modelling the macroscopic behaviour of composite materials. These solutions have very oscillating behaviour and therefore couldn't be obtained by standard numerical computations. This work is important for fundamental research as well as for numerous applications, especially in the mechanics of composite materials.

Here we study deformation of a breather type solution for the linear differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + ib\frac{\partial^3 u}{\partial t^2 \partial x} + \frac{\partial^4 u}{\partial t^2 \partial x^2}, \quad |b| < 2, \quad (1)$$

with discontinuous initial data:

$$u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0, \end{cases} \qquad u_t(x,0) = 0. \quad (2)$$

The problem arises in the study of wave motion in periodic stratified media [1, 2].

In [3] the existence of the breather type solution:

$$\text{at} \quad t \to \infty, \quad |x| < ct^{-1/2}$$

$$u(x,t) = \frac{1}{2} + \frac{\text{sgn}(x)}{2}\cos(t) + \text{O}(\sqrt{x^2 t})$$

had been proved considering the equation (1) for $b = 0$. Numerical experiments have confirmed the existence of a solution with exotic (for a linear equation) asymptotics [4].

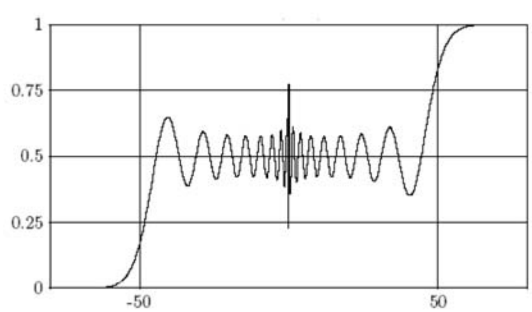

Figure 1: $b = 0$, Re $u(x, 50), le = 80$

The breather stands out against oscillations of a smaller amplitude (see Fig.1). These oscillations are located between the characteristics $x \pm t = 0$. Outside the region between the characteristics, the solution tends exponentially rapidly to the limit values 0 when $x \to -\infty$ and 1 when $x \to +\infty$.

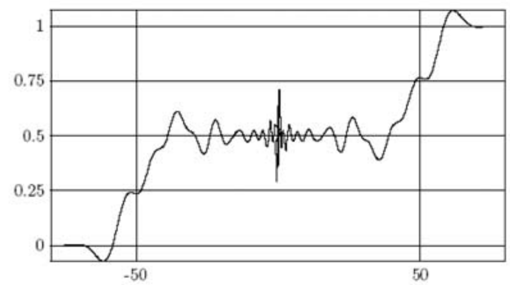

Figure 2: $b = 1$, Re $u(x, 50), le = 80$

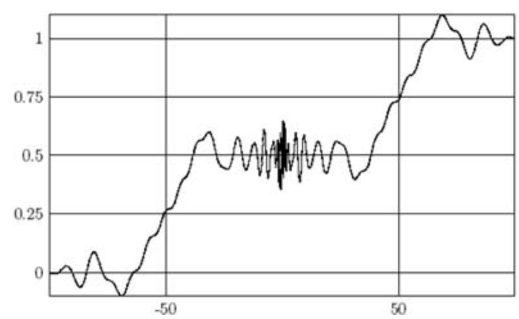

Figure 3: $b = 1.5$, Re $u(x, 50), le = 100$

The numerical experiments performed have shown the existence of breather-type solutions in the case of $b \neq 0$ as well.

Figures 2 and 3 show the real part of the solution of the problem (1),(2) at $t = 50$ for $b = 1$ and $b = 1.5$, respectively.

Numerically, instead of the Cauchy problem we solved an initial-boundary value problem inside the interval $|x| \leq le$.

When t is big enough, a difference between the solution in the case of $b = 0$ and the real part of the solution in the case $b \neq 0$ appears. Even at $t = 9$ (in the case $b = 1$), steps near characteristics are outlined. When $b \neq 0$ the oscillation amplitude decreases, while the region of the oscillations broadens. An exponentially fast recovery of the limit values happens outside the region between the characteristics.

In [5] we proved asymptotics when $t \to \infty$ for the problem (1-2) in the case $b = 1$. They confirm the validity of the breather deformation processes detected in numerical simulation. Specifically, it is proved that the support of the breathes is reduced to $|x| < c/t$ (against $|x| < c/\sqrt{t}$ in the case $b = 0$) and the exponential decay to the limiting values occurs at $|x| = 1.215...t$ (against $|x| = t$ in the case $b = 0$). As b increases, the oscillation zone expands and in the limit ($|b| = 2$ it occupies the entire real line. The boundaries of the oscillation zone are easily calculated via the multiple critical points of the function $s/\sqrt{1 - bs + s^2}$. The asymptotics are proved by using methods of complex variable methods - stationary phase method and saddle point method, particularly [6]. Beforehand, following [7], the solution to the problem (1-2) is represented in the form of the contour integral

$$u = -\frac{1}{4\pi i} \int\limits_{\Gamma} \Bigg(\exp(-\frac{ist}{\sqrt{1 - bs + s^2}} - ixs) + \exp(\frac{ist}{\sqrt{1 - bs + s^2}} - ixs) \frac{ds}{s}.$$

The contour Γ goes along the real line except a neighborhood of zero: the pole is rounded in the upper half plane over a semicircle of small radius.

References

[1] N.S. Bakhvalov and M.E. Eglit, Dokl.Math. 61, 1-4 (2000)[ДАН. 2000. Т.370. No.1. С.7-10].

[2] N.S. Bakhvalov and M.E. Eglit, Dokl.Math. 65,301-305 (2002)[ДАН. 2002. Т.383. No.6. С.742-746].

[3] S.I. Serdyukova, Dokl.Math. 67, 203-207 (2003) [ДАН. 2003. Т.389. No.3. С. 305-309].

[4] S.I. Serdyukova, Program. Comput. Software 30, 118-119 (2004) [ПРОГРАММИРОВАНИЕ. 2004. Т.389. No.2. С. 79-80].

[5] S.I. Serdyukova, Dokl.Math. 80, 1-7 (2009) [ДАН.2009. Т.427. No.1. С. 17-23].

[6] M.V. Fedoryuk, *Asymptotics: Integrals and Series*(Nauka, Moscow, 1987)[in Russian].

[7] M.V. Fedoryuk, Mat.Sb. 62(104), 397-486 (1963).