Punctuated Evolution due to Delayed Carrying Capacity

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Most natural and social systems evolve according to multistep processes. We refer to this kind of dynamics as *punctuated evolution*, because it describes the behavior of nonequilibrium systems that evolve in time, not according to a smooth or gradual fashion, but by going through periods of stagnation interrupted by fast changes. These include the growth of urban population, the increase of life complexity and the development of technology of human civilizations, and, more prosaically, the natural growth of human bodies.

According to the theory of punctuated equilibrium, the evolution of the majority of sexually reproducing biological species on Earth also goes through a series of sequential growth-stagnation stages. The resulting punctuated-equilibrium concept of the evolution of biological species is well documented from paleontological fossil records.

The development of human societies provides many other examples of punctuated evolution. For instance, governmental policies, as a result of bounded rationality of decision makers, evolve incrementally. The growth of organizations, of firms, and of scientific fields also demonstrates nonuniform developments, in which relatively long periods of stasis are followed by intense periods of radical changes. During the training life of an athlete, sport achievements rise also in a stepwise fashion. There are many other examples.

Despite these ubiquitous empirical examples of punctuated evolution occurring in the development of many evolving systems, to our knowledge, there exists no mathematical model describing this kind of evolution.

A new delay equation is introduced to describe the punctuated evolution of complex nonlinear systems. It is surprisingly rich in the variety of regimes that it describes, depending on the system parameters. In addition to the process of punctuated increase, it demonstrates punctuated decay, punctuated up-down motion, effects of mass extinction, and finite-time catastrophes.

Detailed analytical and numerical investigations provide the classification of all possible types of solutions for the dynamics of a population in the four main regimes dominated respectively by: (i) gain and competition, (ii) gain and cooperation, (iii) loss and competition, and (iv) loss and cooperation. Our delay equation may exhibit bistability in some parameter range, as well as a rich set of regimes, including monotonic decay to zero, smooth exponential growth, punctuated unlimited growth, punctuated growth or alternation to a stationary level, oscillatory approach to a stationary level, sustainable oscillations, finite-time singularities as well as finite-time death.

General model

Consider the delay evolution equation for a normalized measure characterizing the population development

$$\frac{dx}{dt} = \sigma_1 x \; - \; \sigma_2 \, \frac{x^2}{y} \; , \qquad (1)$$

where $y = y(t) = a + bx(t-\tau)$ is a dimensionless carrying capacity, and time lag $\tau \geq 0$.

This equation is complemented by an initial history condition

$$x(t) = x_0 \quad (t \leq 0) \; , \qquad (2)$$

according to which $y(t) = y_0 = a + bx_0$ for $t \leq 0$.

- The coefficient a, characterizing the initial resources provided by Nature, is non-negative.
- The coefficient b, controlling the impact of past population on the present carrying capacity, can be either positive or negative, depending on whether production or destruction dominates. A known example for $b < 0$ is the destruction of habitat by humans, associated with deforestation, reduction of biodiversity, and climate changes. The destruction of the global Earth ecosystem is caused by the rapid growth of the human population, which is sometimes compared with a pathological cancer process that could result in the eventual extinction of the human population. Another example of destructive activity is firm mismanagement, and operational risks, which can result in firm bankruptcy and even in a global economic crisis, when many economic and financial institutions are mismanaged. One more illustration is the destruction of the economy of a country by a corrupted government. In contrast, a positive b corresponds to improved exploitation of resources and increased productivity.

- The initial value x_0 of the dimensionless population is positive.

- The initial value y_0 of the carrying capacity can be either positive or negative. The standard case is, of course, $y_0 > 0$. A negative value y_0 of the effective carrying capacity at $t = 0$ can be interpreted as describing a strongly destructive action of the agents that occurred in the preceding time interval $[-\tau, 0]$.

The above statements translate into

$$a \geq 0 \; , \qquad -\infty < b < \infty \; ,$$

$$x_0 > 0 \; , \qquad -\infty < y_0 < \infty \; .$$

We restrict our investigation to non-negative dimensionless population size $x(t) \geq 0$.

There are thus four possible types of societies, depending on the signs of σ_1 and σ_2:

$$\begin{aligned} &\sigma_1 > 0 \;\&\; \sigma_2 > 0 \quad (\text{gain} + \text{competition}), \\ &\sigma_1 > 0 \;\&\; \sigma_2 < 0 \quad (\text{gain} + \text{cooperation}), \\ &\sigma_1 < 0 \;\&\; \sigma_2 > 0 \quad (\text{loss} + \text{competition}), \\ &\sigma_1 < 0 \;\&\; \sigma_2 < 0 \quad (\text{loss} + \text{cooperation}). \end{aligned} \tag{3}$$

Here we present the summary of the case of prevailing gain, when $\sigma_1 > 0$, and competition, when $\sigma_2 > 0$. Detail description of this and other variants described by conditions (3) can be found in Ref. [1].

When gain (birth) prevails over loss (death) and competition prevails over cooperation, this corresponds to the first line in the classification (3). Then Eq. (1) translates into

$$\frac{dx(t)}{dt} = x(t) \; - \; \frac{x^2(t)}{a + bx(t-\tau)} \; . \tag{4}$$

In the general case, there are two stationary solutions

$$x_1^* = 0 \; , \quad x_2^* = \frac{a}{1-b} \; . \tag{5}$$

The first fixed point, x_1^*, is unstable for any $a > 0$ and any b, and all $\tau > 0$. The second fixed point x_2^* is stable in one of the regions, when either

$$a > 0 \; , \quad -1 < b < 1 \; , \quad \tau \geq 0 \; ,$$

or

$$a = 0 \; , \quad 0 < b < 1 \; , \quad \tau \geq 0 \; ,$$

or

$$a > 0 \; , \quad b < -1 \; , \quad \tau < \tau_0 \; ,$$

where

$$\tau_0 \equiv \frac{1}{\sqrt{b^2 - 1}} \; \arccos\left(\frac{1}{b}\right) \; .$$

The point x_2^* becomes a stable center (associated with a vanishing Lyapunov exponent λ_2) for $a > 0$, $b < -1$, $\tau = \tau_0$. The value τ_0 diverges, if $b \nearrow -1$, as

$$\tau_0 \simeq \frac{\pi}{\sqrt{2(|b| - 1)}} \quad (b \nearrow -1) \; .$$

Varying the system parameters yields the different solutions, which we analyze successively.

Punctuated unlimited growth

When the carrying capacity increases, due to the intensive creative activity of the agents forming the system, which corresponds to the parameters

$$a \geq 0 \; , \quad b \geq 1 \; , \quad \tau \geq 0 \; ,$$

then $x_0 < y_0$ and the fixed point x_2^* does not exist. The function $x(t)$ grows by steps of duration $\simeq \tau$, tending to infinity as time increases to infinity.

Punctuated growth to a stationary level

For a lower creative activity (quantified by b) of the population affecting the effective carrying capacity, i.e., for

$$a > (1-b)x_0 \; , \quad 0 \leq b < 1 \; , \quad \tau \geq 0 \; ,$$

which implies that $x_0 < y_0 < x_2^*$. the value of $x(t)$ monotonically grows by steps to the stationary solution x_2^*

Punctuated decay to a stationary level

When the pre-existing carrying capacity a is smaller than in the previous cases and the creation coefficient b is not too high, so that

$$0 \leq a < (1-b)x_0 \; , \quad 0 \leq b < 1 \; , \quad \tau \geq 0 \; ,$$

which means that $x_0 > x_2^* > y_0 > 0$, then $x(t)$ monotonically decays by steps to the stationary solution x_2^*.

Punctuated alternation to a stationary level

When the initial capacity a is large, but the agent activity is destructive, with the parameters

$$a > |b|x_0 \; , \quad -1 \leq b < 0 \; , \quad \tau \geq 0 \; ,$$

there are two subcases. If $a > (1+|b|)x_0$, so that $x_0 < x_2^* < y_0$, then $x(t)$ grows initially. And if $|b|x_0 < a < (1+|b|)x_0$, so that $x_0 > x_2^* > y_0 > 0$, then $x(t)$ decreases initially. However, the following behavior in both these subcases is similar: $x(t)$ tends to the stationary solution x_2^* through a sequence of up and down alternations.

Oscillatory approach to a stationary level

If the capacity is large and the destructive activity is rather strong, such that

$$a > |b|x_0 \; , \quad b < -1 \; , \quad \tau < \tau_0 \; ,$$

there are again two subcases, when $x(t)$ either increases or decays initially. But the following behavior for both these subcases is again similar: $x(t)$ tends toward the focus x_2^* by oscillating around it. Contrary to the previous case, here the stagnation stages are practically absent, so that the overall evolution is purely oscillatory, with a decaying amplitude of oscillations.

Everlasting nondecaying oscillations

With the parameters a and b as in the previous case, but with the time lag being exactly equal to τ_0, when

$$a > |b|x_0 \; , \quad b < -1 \; , \quad \tau = \tau_0 \; ,$$

then $x(t)$ oscillates around the center x_2^* without decaying. At the initial time, x can either increase or decrease, as in the previous cases. But, it will rapidly set into a stationary oscillatory behavior without attenuation.

Punctuated alternation to finite-time death

The fact that the behavior of the system depends sensitively on the time lag τ is well exemplified by the regime in which the values of a and b are the same as in regime for everlasting nondecaying oscillations, but the lag becomes longer, so that

$$a > |b|x_0 \; , \quad b < -1 \; , \quad \tau > \tau_0 \; .$$

In this regime, $x(t)$ alternates between upward and downward jumps, with increasing amplitude, until it hits the zero level at a finite death time t_d defined by the equation $a + bx(t_d - \tau) = 0$, at which time the rate of decay becomes minus infinity. As in the previous cases, depending on whether $x_0 < y_0$ or $x_0 > y_0$, the initial motion can be either up or down, respectively. But the following behavior follows a similar path, with $x(t)$ always going to zero in finite time. The abrupt fall of the population $x(t)$ to zero can be interpreted as a *mass extinction*, as has occurred several times for species on the Earth. The effect of mass extinction in the present example is caused by the intensive destructive activity ($b < -1$) of the agents composing the system. This is an example of total collapse caused by *the destruction of habitat.*

Growth to a fixed finite-time singularity

Another example of catastrophic behavior happens when the initial carrying capacity is negative ($y_0 < 0$). This occurs when the habitat has been destroyed in the preceding time interval $[-\tau, 0]$ and the destruction goes on for $t > 0$. For the set of parameters

$$a < |b|x_0 \; , \quad b < 0 \; , \quad \tau \geq t_c \; ,$$

with a sufficiently long time lag τ, the function $x(t)$ diverges at the singularity time t_c, given by the expression $t_c = \ln(1 - y_0/x_0)$. The divergence is hyperbolic, i.e., in the vicinity of t_c, $x(t) \simeq y_0/(t_c - t)$ for $t \to t_c - 0$. For the considered parameters a, b, the singularity always occurs at the critical time t_c determined by the values of x_0 and y_0, independently on the delay time τ as long as τ is larger than t_c.

Growth to a moving finite-time singularity

When the delay time τ is smaller than the singularity time $t_c = \ln(1 - y_0/x_0)$, and

$$a < |b|x_0 \; , \quad b < 0 \; , \quad \tau_c < \tau \leq t_c \; ,$$

the critical lag τ_c can only be determined numerically. In this regime, $x(t)$ grows without bound and reaches infinity in a finite time at a moving singularity time $t_c^* \geq t_c$ which is a function of τ. We find that t_c^* goes to infinity as τ decreases to τ_c.

The catastrophic divergence of $x(t)$ can be interpreted as a diagnostic of a transition to another state or to a different regime in which other mechanisms become dominant. It is natural to interpret the critical points as periods of transitions to new regimes.

Exponential growth to infinity

As the delay time τ becomes smaller than the threshold value τ_c, defined in the previous section, i.e., for the following parameters

$$a < |b|x_0 \; , \quad b < 0 \; , \quad 0 < \tau \leq \tau_c \; ,$$

the finite-time singularity does not exist anymore. The function $x(t)$ exhibits a simple unbounded exponential growth to infinity, as time tends to infinity.

The exact limit of a zero time delay $\tau = 0$ is not included in this regime. When τ is exactly zero, the exponential growth regime is replaced abruptly into the regime with a fixed-time singularity.

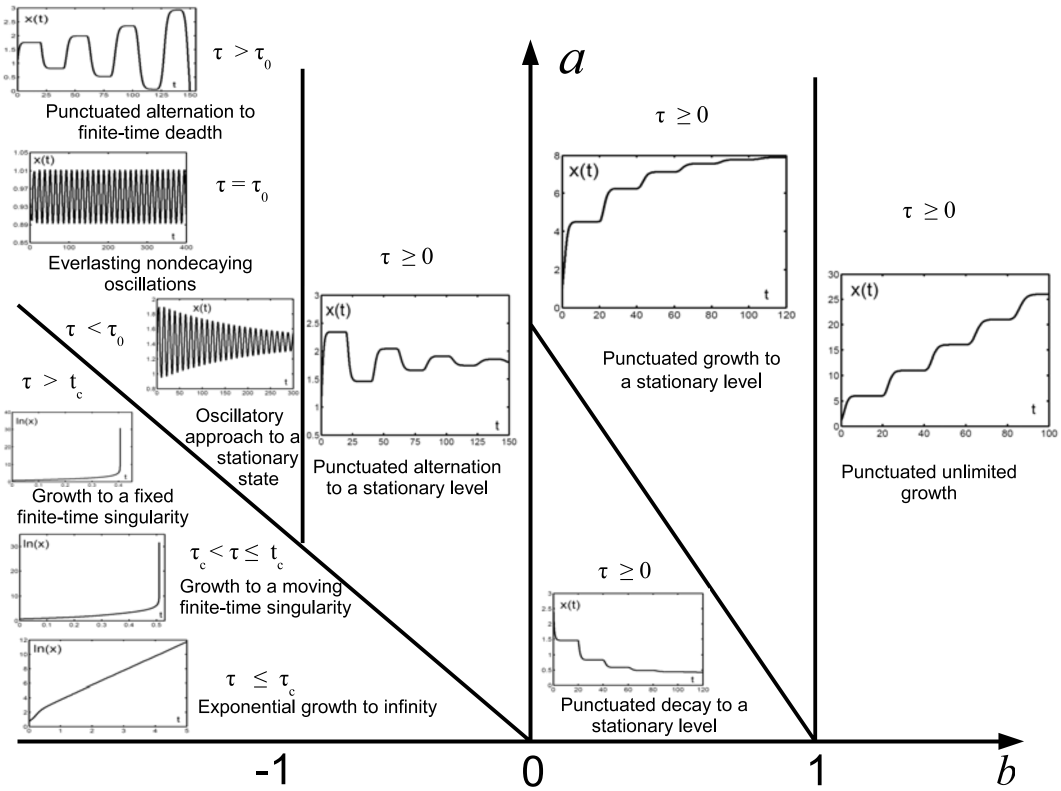

Figure 1: Scheme of the variety of qualitatively different solution types for the most complicated and the most realistic regime, when gain (birth) prevails over loss (death) and competition is stronger than cooperation.

Conclusion

We have carefully investigated all the possible emergent regimes, using both analytical and numerical methods. This has led to a complete classification of the possible types of different solutions. It turns out that there exists a large variety of solution types. In particular, we find a rich and rather sensitive dependence of the structural properties of the solutions on the value of the delay time τ. For instance, in the regime where loss and competition are dominant, depending on the value of the initial carrying capacity and of τ, we find monotonic decay to zero, oscillatory approach to a stationary level, sustainable oscillations and moving finite-time singularities. This should not be of too much surprise, since delay equations are known to enjoy much richer properties than ordinary differential equations. In this spirit, Kolmanovskii and Myshkis [2] provide an example of a delay-differential equation, whose properties are as rich as those of a system of ten coupled ordinary differential equations. We have illustrated in different figures the main qualitative properties of the different solutions, not repeating the presentations of solutions with similar behavior.

Figure 1 presents the scheme of the variety of qualitatively different solution types described above for the most complicated and the most realistic regime, when gain (birth) prevails over loss (death), $\sigma_1 > 0$, and competition is stronger than cooperation, $\sigma_2 > 0$. Detailed description and investigation of all possible regimes enumerated in (3) can be found in Ref. [1].

References

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[2] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Kluwer, Dordrecht, 1999.