Incoherent Tunneling Helps to Solve Intractable Problem

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Abstract. Theory of computer calculations strongly depends on the nature of elements the computer is made of. Quantum interference allows to formulate the Shor factorization algorithm turned out to be more effective than any one written for classical computers. Similarly, quantum wave packet reduction allows to devise the Grover search algorithm which outperforms any classical one. In the present paper we argue that the quantum incoherent tunneling can be used for elaboration of new algorithms able to solve some NP-hard problems, such as the Traveling Salesman Problem, considered to be intractable in the classical theory of computer computations.

Usually only computations that do not use resources that grow exponentially are of interest (so-called *efficient* computations). For them the *Strong Church's Thesis* was formulated [1]: Any finite analog computer can be simulated *efficiently* by a digital computer, in a sense that the time required by the digital computer to simulate the analog computer is bounded by a polynomial function of the resources used by the analog computer. Due to a great progress in development of digital computer techniques, analog computers are used now extremely seldom. Nevertheless, they survived in a virtual form inside digital computers as analog algorithms and turned into powerful heuristic methods for solution of optimization problems. The analog algorithms inherited from their natural ancestors the correspondence of computational schemes to some real physical processes in nature. Thus, the *steepest descent* method is based on gradient equations often used for far-from-equilibrium physical system description. It is used to find local minima. *Simulated annealing* describes the opposite situation when a physical system is very close to the equilibrium state at each moment of its slow dynamical evolution. It is experimentally discovered and theoretically understood that molten matter, to be subjected to slowly cooling, transforms into a crystalline state corresponding to the minimum of its free energy. However, rigorous consideration showed that in fact the global minimum can never be reached, because simulated annealing with cooling schedule

$$T_1,\ T_2,\ ...,\ T_n \to 0,$$

requires an exponential computation time:

$$t_n \sim \exp(A/T_n), \qquad A = const.$$

Even a very simple minimization problem corresponding to a biased double well energy function shown in Fig.1 requires the cooling schedule

$$T_n \sim D/\ln\ n$$

and an exponential computational time, $t \sim \exp n$, and, therefore, is intractable in the frame of this algorithm.

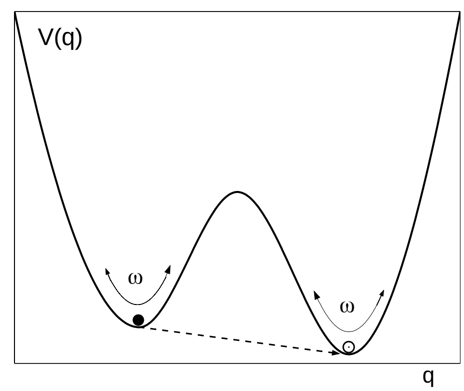

Figure 1: Minimization problem corresponding to a biased double well energy function requires the cooling schedule $T_n \sim D/\ln n$ and an exponential computational time in the frame of the simulated annealing algorithm.

For quantum computations no statements similar to the Strong Church's Thesis were formulated so far and, therefore, both analog and digital approaches are still of equal interest. Below, we discuss an approaches to solution of the Traveling Salesman Problem by a quantum analog computing machine.

Now we give plausible reasoning in favor of incoherent tunneling as an effective remedy for solution of minimization problems on the stage of low temperatures when the simulated annealing algorithm is inoperative. To this goal let us consider again a biased double well potential shown in Fig.2 and a particle localized initially in the left local minimum at point q_0. Due to quantum tunneling effect, the particle can penetrate the potential barrier and fall into the global minimum at q_1. When the motion of the particle is accompanied by dissipation, there is a chance that the particle will stay at the global

[1]Detailed references for all facts noticed in this paper can be found in [1].

minimum and, hence, an optimization problem will be resolved.

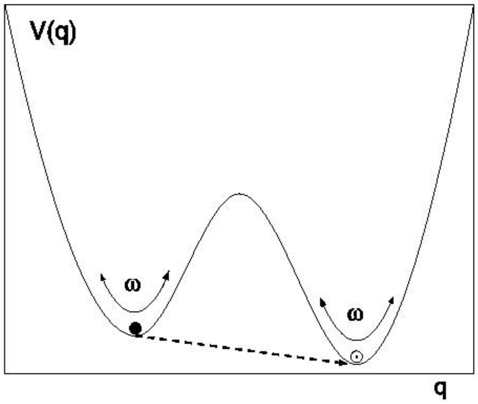

Figure 2: Tunneling with dissipation allows particle to penetrate through the potential barrier in a finite space of time and then to stay at the point of the global minimum.

In fact, a plausible picture sketched above may be rigorously grounded. If thermal energy, $k_B T$, is much less than energy, $\hbar\omega$, of classical oscillations at the positions of minima and the latter is, in its turn, much than the height of the potential barrier, V_{bar}, between the wells,

$$k_B T \ll \hbar\omega \ll V_{bar},$$

the process of tunneling with dissipation can be described on the base of multi-instanton calculations in the framework of the imaginary time functional integral approach. For temperature close to zero, the tunneling rate from q_0 to q_1 is given by an expression:

$$\gamma(q_0 \to q_1) = \frac{\pi}{2} \frac{\triangle^2}{\omega} \frac{1}{\Gamma(2\alpha)} \left(\frac{\sigma}{\omega}\right)^{2\alpha-1},$$

while the inverse transitions are suppressed,

$$\gamma(q_1 \to q_0) = 0.$$

Here $\alpha = \eta d^2/2\pi\hbar$ is dimensionless damping coefficient, $d = q_1 - q_0$ is the tunneling length, η describes the force of friction, $F = -\eta dq/dt$, and $\triangle$ is the tunnel matrix of undamped and unbiased system. The time of transition to the minimum, τ, is *finite* now:

$$\tau \sim \frac{\hbar}{\gamma}.$$

Thus, at least for the double well, tunneling with dissipation, or incoherent tunneling, can be effectively applied for solving the minimization problem in the vicinity of zero temperatures.

In the case of a tilted periodic potential, displayed in Fig.3, it is also possible to fulfil calculations explicitly and show that the time of transition from

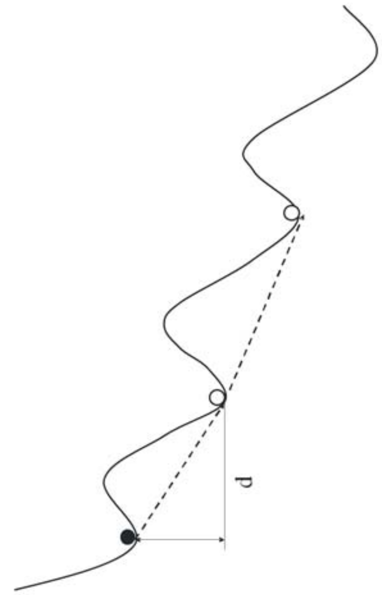

Figure 3: Particle moves down the tilted periodic potential hill with a constant speed.

one local minimum to another is finite. An average position of the particle may be expressed as

$$\langle q(t) \rangle = v\, t,$$

where v is mean drift velocity,

$$v = \frac{d\hbar}{\gamma} \tanh(\beta\hbar\sigma/2), \qquad \beta = 1/k_B T.$$

Therefore, the time of minimization is proportional now to the sum of the tunneling time through all potential barriers on the way to the global minimum. It is naturally to suggest that a similar picture remains in the general case when the slope of the "hill" is variable. Thus, the main question left to be discussed is how many local minima actually are on the way to the global minimum. In the case of the exponential number of them the problem would remain open.

The practice of solving optimization problems shows that usually there are much less of energy local minima than non-minimal values of energy (for review including the Hopfield Network, Potts-glass with biased patterns, spin-glass models belonging to the NP-hard problems see [1]). For example, numerical experiments have shown the total number of local minima increases as some *small power* of n, rather than $\exp(n)$ for infinite-ranged models of spin-glasses considered by Kirkpatrick.

The Traveling Salesman Problem (TSM-problem) is formulated as follows: given positions of cities, what is the shortest tour in which each city is visited

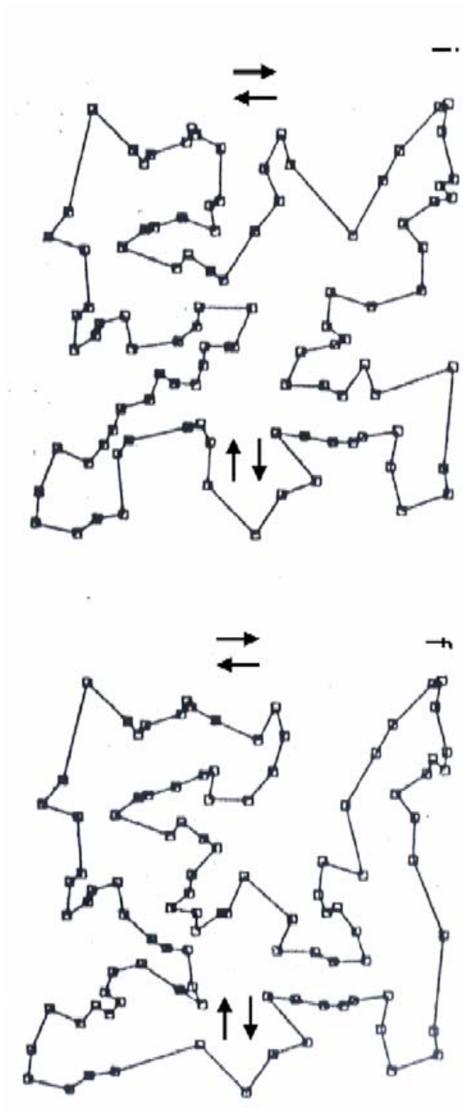

Figure 4: The process of solution of TSM-problem by the domain wall analog computer: an input, i, corresponding to a local minimum spontaneously decays into the final state, f, describing the global minimum.

once? It is evident that the total number of possible tours increases exponentially with the increase of number, n, of cities:

$$N_{tours} \sim e^{C\,n},$$

and their sequential consideration requires computing time which increases faster than any power of n. Therefore problem is considered as intractable in the classical theory of computation. Now we'll show how the problem can be solved, at least in principle, using the quantum calculation technique.

Quantum analog computers for solving TSM-problem can be constructed on the base of the Elastic Net analog algorithm elaborated for the usual digital computers. The algorithm is grounded on a discrete form of the gradient equation,

$$\frac{d\vec{y}_j}{dt} = -\frac{\partial F}{\partial \vec{y}_j},$$

with free energy, F:

$$F = -\alpha k^2 \sum_i \ln \sum_j e^{-|\vec{x}_i - \vec{y}_j|/2k^2} + \beta k \sum_j |\vec{y}_{j+1} - \vec{y}_j|^2.$$

For $\triangle t = 1$ it is possible to write:

$$\triangle \vec{y}_j \simeq -\frac{\partial F}{\partial \vec{y}_j},$$

$$\triangle F \simeq \sum_j \frac{\partial F}{\partial \vec{y}_j} \triangle \vec{y}_j \simeq -\sum_j \left(\frac{\partial F}{\partial \vec{y}_j}\right)^2 < 0.$$

This means that the algorithm directs the system to a local minimum of F in accordance with the steepest descent method. The explicit form of F can be written using a very clear physical picture. Positions of cities are described by $\vec{x}_i$, points with coordinates $\vec{y}_j$ lie on an elastic string. Each point moves under the influence of two types of force. The first moves it towards those cities to which it is nearest; the second pulls it towards its neighbors on the string, acting to minimize the total string length.

Elastic string can be created as a physical computation device in the form of a mesoscopic domain wall in an antiferromagnetic thin film at low temperature. Tunneling of the domain walls in ferromagnetic and antiferromagnetic insulators through potential barriers is described in a similar way as incoherent tunneling of particles, considered above. The role of the barriers can play defects or other singular points in the lattice. The domain wall has a variety of dissipative couplings to the magnons, photons, impurities, defects, and phonons. Nevertheless, it was shown that large domain walls, containing up to 10^{10} spins can behave as quantum objects at low temperatures. Both theory and experiment revealed a finite tunneling time of domain walls through potential barriers. This gives an opportunity of fabrication of an analog quantum computer using the incoherent tunneling effect. The process of solution of TSM-problem by this computer is shown schematically in Fig.4. Firstly, one creates an input, i, corresponding to some local minimum found with the help of the classical computer, and then, after a time, obtains a final state, f, as the solution. Points in the picture denote "cities", which are some kind of defects in the lattice that pin the domain wall to necessary locations. They may be some implanted atoms with high value of spins, or strongly magnetized atoms. Tension of the string can be regulated by a change of the mutual orientation of directions of spins on either side of the domain wall. It is clear that a value of tension should be small enough to prevent the string from a temptation to pass round some cities to minimize its length. This means that directions of spins on either side of the domain wall should not be much differing. Such a slack string limit exactly corresponds to the strong shot-range interaction between the string and the cities considered as the last step of the iteration procedure in the classical Elastic Net analog algorithm.

There is a commonly accepted method to demonstrate an efficiency of a new quantum algorithm with the help of a classical computer simulation of corresponding quantum calculations. But in our case there is no need to do so, because this has been *already* done. Actually, some kind of incoherent tunneling was used efficiently many times as an optimizing procedure during simulated annealing. For instance, Kirkpatrick suggested to check not only neighboring (in the Hamming sense) configuration of spin during minimization, but also states which were 2, 3 and 4 steps away (for details see [1]). When energy of a trial state was found to be less than energy of the current state, the system was transmitted to the trial state. This corresponds to the under-barrier transitions through the potential barriers with 1, 2 and 3 Hamming's steps of width. Of course, such a strategy is hampered for classical computers for more wider potentials barriers by a huge number of possible trial states. Quantum incoherent tunneling turns out to be much more effective, because it allows to run over all local energy minima only, without examination of all possible values of energy for all possible trial states.

Thus, setting aside purely engineering problems concerned with creation of the domain wall described above, one may conclude that the Traveling Salesman Problem can be solved, at least theoretically, by a quantum analog computer in many practically important cases (when the number of local minima is described with a polynomial function). Our consideration has also shown that investigation of the number of local minima of free energy in the vicinity of the global one is a general mathematical problem of great importance. A drastic decrease of the number of local minima in the case under consideration follows from general reasoning based on the Nernst theorem for entropy,

$$\lim_{T \to 0} S(T) = 0,$$

if there are no gaps of function $S(T)$ near $T = 0$. In principle, a gap is possible if a phase transition of the first kind exists in the limit $T \to 0$.

We are grateful to V.V Ivanov, who attracted our attention to the classical Elastic Net algorithm and to M.Z. Yuriev, who insisted on necessity of quantum analog algorithms investigation.

References

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