Geodesic Flows on the Energy Hypersurface of the Three-Body System

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Abstract. The reduction of the three-body classical problem is considered in the framework of the ideas of separation of the internal and external motions of a body-system. Based on the fact that for Hamiltonian system there exists equivalence between phase trajectories and geodesics trajectories on the Riemannian manifold (the energy hypersurface of a body-system), the classical three-body problem is formulated in the framework of six geodesic equations. It is shown that in the case when the total interaction potential of a body-system depends on the relative distances between particles, the three from six geodesics equations describing rotations of formed by three bodies triangle are solved exactly. Using this fact the problem of three-body is possible to describe by three nonlinear ODEs of a canonical form that in the phase space is equivalent to autonomous system of the sixth-order. It is shown that the reduced problem describes the dynamics of the three-body system on the scattering plane with consideration of the total angular momentum of the rotating body-triangle. The system of algebraic equations for finding of homographic solutions of restricted three-body problem is obtained.

Introduction

The general three-body classical problem concerns the question of understanding the motions of three arbitrary point masses traveling in space according to Newton's laws of mechanics. Many works on analytical mechanics, celestial mechanics, stellar and molecular dynamics (see [1, 2, 3, 4, 5, 6, 7, 8]) are devoted to the study of this problem. For solution of the general problem different approaches based on series expansions methods have been proposed; however, due to the poor convergence of these expansions they are often used and are useful only for solving of particular problems where the system of three-bodies is in a stable bound state [1, 9]. Moreover, the three-body problem is a typical example of a dynamic system where on the large scales of the phase space we observe all features of a complex motion including the bifurcation and chaos. That makes the numerical simulation method a basic way of research of the mentioned problem.

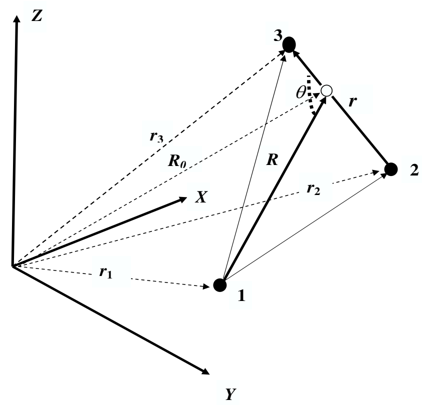

Figure 1: *The Cartesian coordinates system where the set of vectors* $\boldsymbol{r}_1, \boldsymbol{r}_2$ and $\boldsymbol{r}_3$ *denotes coordinates of the 1st, 2nd and 3rd particles, respectively. The* ◯ *is the center-of-mass of pair* (12) *which in the Cartesian system is expressed by* $\boldsymbol{R}_0$. *The Jacobi coordinates system described by the radius-vectors* $\mathbf{R}$ *and* $\boldsymbol{r}$, *in addition to* θ, *denote scattering angle.*

The main aim of this work is finding new opportunities to separation of the internal and external motions in the general classical three-body problem. The last will have a key importance for the reducing of dimensionality of dynamical problem that will allow to develop an effective algorithm for numerical simulation.

The classical three-body system in the laboratory frame

The classical Hamiltonian of the three-body system after Jacobi [9] and mass-scale [10] transformations can be written in the form (see also [11]):

$$\mathrm{H}(\mathbf{r};\mathbf{p}) = \frac{\mathbf{p}^2}{2\mu_0} + \mathrm{V}(\mathbf{r}), \tag{1}$$

where $\mathbf{r} = \boldsymbol{r} \oplus \boldsymbol{R} \in R^6$ and $\mathbf{p} \in R^6$ correspondingly the position vector and the momentum of the effective mass (imaginary point) $\mu_0 = [m_1 m_2 m_3/(m_1 + m_2 + m_3)]^{1/2}$, ($m_1, m_2$ and m_3 masses of bodies). Note that $\boldsymbol{r}$ designates the distance between 2 and 3 bodies, while $\boldsymbol{R}$ is the distance between $1th$ particle and the center of mass of the pair (2,3).

Without going into details note that the considered problem has 12 integrals of motions using which the initial $18th$ order system to the $8th$ order system is reduced [12].

Geodesic equations in conformal-Euclidean space

As it is easy to see, the classical system of three bodies at motion in the $3D$ Euclidean space permanently forms a triangle, and Newton's equations describes the dynamical system on the space of such triangles [13]. This means that we can formally consider the motion of a body-system consisting of two parts. The first is the rotational motion of the body-triangle in the $3D$ Euclidian space and the second is the internal motion of bodies on the plane defined by the triangle. Mathematically, the configuration manifold of solid body R^6 can be represented as a direct product of two subspaces [14]:

$$R^6 \cong R^3 \times S^3,$$

where R^3 is the manifold which is defined as an orthonormal space of relative distances between bodies while S^3 denotes the space of the rotation group $SO(3)$. However in the considered problem, connections between of bodies are not holonomic and respectively we must change the representation for the configuration manifold $\mathbf{M}$.

Let us consider the region of changing of coordinates an imaginary point on the plane formed by the system of three-body (further will be named *the internal space* $\mathcal{M}_t$):

$$x^1 = ||\boldsymbol{r}|| \in \big[0,\infty\big), \quad x^2 = ||\boldsymbol{R}|| \in \big[0,\infty\big), \quad x^3 = ||\boldsymbol{r}+\boldsymbol{R}|| = \sqrt{(x^1)^2 - 2x^1x^2\cos\theta + (x^2)^2} \in L, \quad (2)$$

where θ is the angle between the vectors $\boldsymbol{r}$ and $\boldsymbol{R}$ which in the Jacobi coordinates system is the scattering angle, in addition $L = [x^1 - x^2, x^1 + x^2]$. The set of *internal coordinates* $\{\bar{x}\} = (x^1, x^2, x^3) \in \mathcal{M}_t$. The rotation of a plane defined by body-triangle will be described by the set of three *external coordinates* $(x^4, x^5, x^6) \in S^3_t$, where S^3_t is a space of the rotation group $SO(3)$ in a neighborhood of interior points $M_i\{(x^1, x^2, x^3)_i\} \in \mathcal{M}_t$. The subset of all interior points $\breve{\mathbf{M}} \subset \mathbf{M}$ is represented as:

$$\breve{\mathbf{M}} \cong \mathcal{M}_t \times S^3_t.$$

The set $\mathbf{M} \setminus \breve{\mathbf{M}}$ has zero measure however in some cases it can be important for dynamics of the three-body system.

So, we can define a local coordinate system in which will be carried further studies:

$$\overline{x^1, x^6} = \{x\} \in \breve{\mathbf{M}}. \qquad (3)$$

Taking into account the well-known work of Krylov [15], we will study the motion of a three-body system on the hypersurface of potential energy of the body-system. In particularly a metric of the hypersurface can be defined in the form:

$$g_{\mu\nu}(\{x\}) = g(\{x\})\delta_{\mu\nu}, \qquad g^{\mu\nu} = g^{-1}(\{x\})\delta_{\mu\nu},$$
$$g(\{x\}) = \big[E - U(\{x\})\big]U_0^{-1} > 0, \qquad (4)$$

where E and $U(\{x\}) \equiv V(\mathbf{r})$ the total energy and potential of the body-system correspondingly, in addition $U_0 = \max||U(\{x\})||$. In the case when the total interactions potential depend from relative distances between particles for the metric tensor we have the equality $g_{\mu\nu}(\{x\}) \equiv g_{\mu\nu}(\{\bar{x}\})$.

The geodesic equations on the Riemannian manifold can be derived using the variational principle of Maupertuis [14, 16]:

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0, \quad \alpha,\beta,\gamma = \overline{1,6}, \qquad (5)$$

where $\dot{x}^\alpha = dx^\alpha/ds$ and $\ddot{x}^\alpha = d^2x^\alpha/ds^2$; in addition s is a scalar parameter of motion (e.g. the proper time), $\Gamma^\alpha_{\beta\gamma}$ designates Christoffel symbol:

$$\Gamma^\alpha_{\beta\gamma}(\{x\}) = \frac{1}{2}g^{\alpha\mu}\big(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\gamma\mu} - \partial_\mu g_{\beta\gamma}\big), \qquad (6)$$

where $\partial_\alpha \equiv \partial_{x^\alpha}$.

Now using equations (5)-(6) and definition for the metric tensor (4) we can derive the following system of second order ordinary differential equations:

$$\ddot{x}^1 = a_1\Big\{(\dot{x}^1)^2 - \sum_{\mu\neq 1,\,\mu=2}^{6}(\dot{x}^\mu)^2\Big\} + 2\dot{x}^1\Big\{a_2\dot{x}^2 + a_3\dot{x}^3\Big\},$$
$$\ddot{x}^2 = a_2\Big\{(\dot{x}^2)^2 - \sum_{\mu=1,\,\mu\neq 2}^{6}(\dot{x}^\mu)^2\Big\} + 2\dot{x}^2\Big\{a_3\dot{x}^3 + a_1\dot{x}^1\Big\},$$
$$\ddot{x}^3 = a_3\Big\{(\dot{x}^3)^2 - \sum_{\mu=1,\,\mu\neq 3}^{6}(\dot{x}^\mu)^2\Big\} + 2\dot{x}^3\Big\{a_1\dot{x}^1 + a_2\dot{x}^2\Big\},$$
$$\ddot{x}^4 = \dot{x}^4\Big\{a_1\dot{x}^1 + a_2\dot{x}^2 + a_3\dot{x}^3\Big\},$$
$$\ddot{x}^5 = \dot{x}^5\Big\{a_1\dot{x}^1 + a_2\dot{x}^2 + a_3\dot{x}^3\Big\},$$
$$\ddot{x}^6 = \dot{x}^6\Big\{a_1\dot{x}^1 + a_2\dot{x}^2 + a_3\dot{x}^3\Big\}, \qquad (7)$$

where following designations are made:

$$g(\{\bar{x}\}) = g_{11}(\{\bar{x}\}) = ... = g_{66}(\{\bar{x}\}),$$
$$a_i(\{\bar{x}\}) = -(1/2)\partial_{x^i}\ln g(\{\bar{x}\}), \qquad i = 1,2,3.$$

In the system (7), the last three equations are integrated exact:

$$\dot{x}^\mu = J_{\mu-3}/g(\{\bar{x}\}), \quad J_{\mu-3} = const_{\mu-3}, \qquad (8)$$

where $\mu = 4, 5, 6$.

Let us note that J_1, J_2 and J_3 are integrals of motion. Their can be interpreted as projections of the total angular momentum $J = \sqrt{J_1^2 + J_2^2 + J_3^2} = const$ of the three-body system on corresponding axes, which are clearly defined by the initial conditions.

Finally, substituting (8) into equations (7), we obtain the following system of a non-linear second-order ordinary differential equations:

$$\begin{aligned}
\ddot{x}^1 &= a_1\big\{(\dot{x}^1)^2 - (\dot{x}^2)^2 - (\dot{x}^3)^2 - (J/g)^2\big\} \\
&\qquad +2\dot{x}^1\big\{a_2\dot{x}^2 + a_3\dot{x}^3\big\}, \\
\ddot{x}^2 &= a_2\big\{(\dot{x}^2)^2 - (\dot{x}^3)^2 - (\dot{x})^2 - (J/g)^2\big\} \\
&\qquad +2\dot{x}^2\big\{a_3\dot{x}^3 + a_1\dot{x}^1\big\}, \\
\ddot{x}^3 &= a_3\big\{(\dot{x}^3)^2 - (\dot{x}^1)^2 - (\dot{x}^2)^2 - (J/g)^2\big\} \\
&\qquad +2\dot{x}^3\big\{a_1\dot{x}^1 + a_2\dot{x}^2\big\}. \qquad (9)
\end{aligned}$$

Thus the system of equations (9) describes the dynamics of an imaginary point with the effective mass μ_0 on the Riemannian manifold (the hypersurface of the potential energy of the system of three-body); $\mathcal{M} = \big[\{\bar{x}\} \equiv (x^1, x^2, x^3) \in \mathcal{M}_t;\ g_{ij} = \big(E - U(\{\bar{x}\})\big)U_0^{-1}\delta_{ij} > 0\big]$, with taking into account rotation of the body-triangle. The system of equations (9) can be represented as a system of six ODEs of first order:

$$\begin{aligned}
\dot{u} &= 2a_1\big\{u^2 - v^2 - w^2 - (J/g)^2\big\} + 2u\big\{a_2 v + a_3 w\big\}, \\
\dot{v} &= 2a_2\big\{v^2 - u^2 - w^2 - (J/g)^2\big\} + 2v\big\{a_3 w + a_1 u\big\}, \\
\dot{w} &= 2a_3\big\{w^2 - v^2 - u^2 - (J/g)^2\big\} + 2w\big\{a_1 u + a_2 v\big\}, \\
&u = \dot{x}^1, \qquad v = \dot{x}^2, \qquad w = \dot{x}^3, \qquad (10)
\end{aligned}$$

where the first three equations form the system of Riccati equations. Recall that summands type of $(J/g)^2$ in equations (10) describe rotational motion of the effective mass that evidently related to Coriolis forces.

Finally using expression of the metric (4) we can write Hamiltonian of imaginary point with the effective mass μ_0, which executes moving in $6D$ configuration space:

$$\mathcal{H}\big(\{\bar{x}\}; \{\dot{\bar{x}}\}\big) = \frac{1}{2\mu_0} g^{\alpha\beta}(\{\bar{x}\}) p_\alpha p_\beta = \frac{\delta^{\alpha\beta} p_\alpha p_\beta}{2\mu_0 g(\{\bar{x}\})},$$

where $\{\dot{\bar{x}}\} = (u, v, w)$.
Taking into account solutions (8) the Hamiltonian of reduced system can be represented in the form:

$$\mathcal{H}\big(\{\bar{x}\}; \{\dot{\bar{x}}\}\big) = \frac{\mu_0}{2} g(\{\bar{x}\})\Big\{u^2 + v^2 + w^2 + \Big(\frac{J}{g(\bar{x})}\Big)^2\Big\}. \qquad (11)$$

Note that, substituting Hamiltonian (11) into the Hamilton-Jacobi equations and by making simple calculations, we can get geodetic equations (9).

The restricted problem and homographic solutions

An important class of solutions for the restricted three-body problem can be studied without solving of equations system (10). Using equations (10) we can derive conditions at which formation of stable configurations for a three-body system are possible. The first condition which must be satisfied for stable configuration of a body-system is obviously the condition of absence of external forces:

$$\boldsymbol{\nabla}\mathcal{H}\big(\{\bar{x}\}; \{\dot{\bar{x}}\}\big) = 0, \qquad (12)$$

where $\boldsymbol{\nabla} = g^{ij}\partial_j = g^{-1}\sum_{j=1}^{3}\partial_j$.
Substituting (11) into (12) with account of the definition of coefficients a_i we can find the following system of algebraic equations:

$$a_1(\{\bar{x}\}) = 0, \quad a_2(\{\bar{x}\}) = 0, \quad a_3(\{\bar{x}\}) = 0. \qquad (13)$$

Solving the system (13) we can find sets of stationary points $\{\bar{x}\}_i$ where $i = 0, 1...$. It is obvious that from these sets of points stable configurations will form only those for which the following conditions are satisfied:

$$\begin{aligned}
&\partial^2_{11}\mathcal{H}(\{\bar{x}\}_{0i}; \{\dot{\bar{x}}\}_{0i}) > 0, \\
&\det\big(\partial^2_{ij}\mathcal{H}(\{\bar{x}\}_{0i}; \{\dot{\bar{x}}\}_{0i})\big) > 0, \\
&\det\big(\partial^2_{kl}\mathcal{H}(\{\bar{x}\}_{0i}; \{\dot{\bar{x}}\}_{0i})\big) > 0, \qquad (14)
\end{aligned}$$

where $i, j = 1, 2$ and $k, l = 1, 2, 3$; in addition, in the (14) designation $\partial^2_{kl} = \partial^2/\partial x^k \partial x^l$ is made. However, system of equations (13) together with conditions (14) defines stable configurations $(\{\bar{x}\}_{0i}; \{\dot{\bar{x}}\}_{0i} = 0)$ of motionless bodies. Note that these stable stationary configurations are interesting in that they can serve as bases for constructing homographic solutions (the solutions which conserve the configuration of bodies during the time). In other words, near the stationary points $\{\bar{x}\}_i \approx \{\bar{x}\}_{0i}$ configuration of bodies should be moving freely. The latter means that we can ignore the first derivatives in equations (10) and write them in the form of algebraic equations:

$$\begin{aligned}
&u^2 - v^2 - w^2 - \lambda_0 + 2\big(\lambda_{12}v + \lambda_{13}w\big)u = 0, \\
&v^2 - w^2 - u^2 - \lambda_0 + 2\big(\lambda_{23}w + \lambda_{21}u\big)v = 0, \\
&w^2 - u^2 - v^2 - \lambda_0 + 2\big(\lambda_{31}u + \lambda_{32}v\big)w = 0, \qquad (15)
\end{aligned}$$

where $\lambda_0 = \big(J/g(\{\bar{x}\}_{0i})\big)^2 = const_i \geq 0$, in addition the following designations are made:

$$\lambda_{ij} = \lim_{\{\bar{x}\}_i \to \{\bar{x}\}_{0i}} a_j/a_i, \quad \lambda_{ij} = \lambda_{ji}^{-1}, \quad i, j = 1, 2, 3,$$

Solving the system of equations (15), we can find in the general case eight sets of solutions for velocities

$\{\dot{\bar{x}}\}_{0i}^k$, where $k = 1, ..., 8$. The existence of sets of real solutions will mean that for the body-system with account of rotations on Euler angles there are homographic solutions. In the case when there is at least one set of solutions for the system of equations (15), it is important to seek solutions near stationary points with consideration of conditions (14). By these computations, we can find a region in the phase space where the coupled three-body system (123) depending on specific conditions can be in the stable or quasistable equilibrium state.

Conclusion

As has shown Poincare the three-body problem generally is a non-integrable system whereupon the system of bodies in the phase space often demonstrates a chaotic behaviors. It means that the small differences in the initial conditions produce very great changes in the motion of the system on a relatively smallish intervals of time that practically makes impossible the prediction of evolution of bodies system in the phase space. The last in turn means that any small error at calculations of the three-body problem in a short time can develop into an enormous mistake. Based on the foregoing, the reduction of the dimensionality of the general classical three-body problem is a mathematical problem of great importance. It should be noted that for the solution to this problem a lot of effort was made, but the maximal possible reduction of dimensionality of the three-body and N-body problem achieved only when the motion of bodies constrained on a plane [17].

We have studied the system of three-body on the configuration space (Riemannian manifold) $\breve{\mathbf{M}} \cong \mathcal{M}_t \times S_t^3$. The problem of motion is formulated as a problem of geodesic flows on the $\breve{\mathbf{M}}$. It is shown that from six nonlinear equations exactly are solved three. This give us possibility to reduce the initial $18th$ order system and to lead it to the autonomous system of $6th$ order. The obtained system consists from the three Riccati equations which are symmetric relative to variables. This enables us to derive a set of algebraic equations by which it is possible find all homographic solutions of three-body system. This enables us to derive the system of algebraic equations by which can be found all homographic solutions of the three-body system.

In end we would like to note, that though the represented approach is not a rigorous proof of the reduction of the general three-body problem (it all depends on how important is the set $\mathbf{M} \setminus \breve{\mathbf{M}}$ for an exact description of dynamic of three-body), nevertheless it is an interesting model for theoretical studies of motion in the non-integrable system. In addition, on the basis of this approach it is possible to develop high-performance algorithm and program package for simulation different complex applied problems.

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