On 2D and 3D Localized Solutions with Nontrivial Topology

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Introduction. Investigation of localized energy distributions described by solutions of nonlinear field equations with nontrivial topological properties is the important approach to nonperturbative field theory. In [1] we propose a classification of localized topological solutions (often called topological lumps, TL) which differs from the standard one but seems to be more instructive, in particular when studying new multidimensional ($D = 2, 3$) particle-like solutions with topological charges.

Historically the first localized solutions with nontrivial topology were skyrmions, found in [2] and used for the description of baryons. In our classification they are topological solitons (TS). More than 10 years later great interest has been drawn to the $2D$ Nielsen-Olesen strings-vortices [3] and to the $3D$ 't Hooft-Polyakov hedgehog-monopoles [4, 5]; these solutions – in our classification – belong to topological defects (TD). Note that the above strings-vortices and hedgehog-monopoles have been discovered during the great solitonic boom of 70's, and that is why they were ascribed to the wide class of solitons (lumps). However we believe that the usage of the same term "a soliton" both for TSs and TDs may turn out misleading in some cases (for an example see the last Section, where TLs in the Standard Model are discussed).

Definitions. Both topological defects, TD, and topological solitons, TS, describe particle-like (extended localized, lumps) distributions of field energy, but they (TDs and TSs) differ in topological properties.

For TSs field distributions in R^D of all fields involved are uniform at space infinity, $R \to \infty$ (see for example Fig.1, where space distribution of 3-component unit Heisenberg field in magnetic soliton is depicted). We consider S^N-valued field with $N = D$; then for TSs topological charge (index) is a mapping degree of the S^N-valued field distribution inside infinite radius ($R = \infty$) sphere S^{D-1}, which is considered – because of constancy of all fields on it – as the single point. The space R^D is compactified by adding this infinite point, and thus soliton maps $R^D_{comp} \to S^N$.

Contrary to TSs Topological Defects are given by S^N-valued field distributions, which are nonuniform at $R = \infty$. Their topological indices are mapping degrees of the $R = \infty$ sphere S^{D-1} to

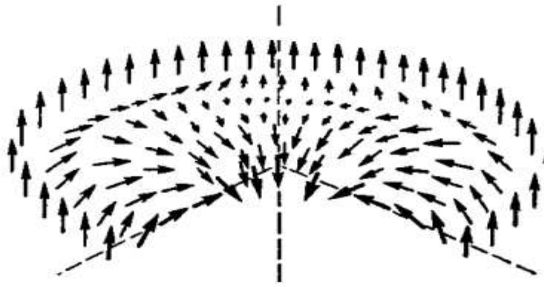

Figure 1: TS with $Q_{top} = 1$ in Heisenberg magnet (for details see [6]).

a S^N sphere defined by the field distribution on this sphere, $S^{D-1} \to S^N, N = D - 1$. From above definitions it is clear that Topological Defects are not Topological Solitons, and vice versa, Topological Solitons are not Topological Defects.

Examples of Topological Solitons and Defects. Now we present some examples of TSs and TDs in D=2 and D=3.

1. Topological Solitons: solitons in Heisenberg magnets ($2D$, $3D$) [6], Belavin-Polyakov solitons/instantons ($2D$) [7], Skyrmions ($3D$) [2], "baby-skyrmions" ($2D$), see, e.g.[8].

2. Topological Defects: sine-Gordon kinks ($1D$), Nielsen-Olesen strings-vortices in the Abelian Higgs model ($2D$) [3], 't Hooft-Polyakov hedgehog-monopoles in the Georgi-Glashow model ($3D$) [4, 5].

Drawbacks of Topological Defects. We believe that TDs have some drawbacks, which are connected with nonuniformity of S^N-valued field at $R = \infty$ (see Fig.2, left). The first of them is that they cannot be generated from unperturbed vacuum state in a finite time. The second one is a problem of matching two (or more) defects. It can be clearly illustrated in two-dimensional ($D = 2$) case for S^1-valued field. In fact, consider two well-separated defects with unit topological charge (see Fig.2, right), so that the center of the first one is located on the x-axis at $x_1 = -\infty$, and the centre of the second one at $x_2 = +\infty$. Then from Fig.2, right one can see that in the vicinity of vertical line $x = 0$ it is impossible to define distribution of S^1 field which is

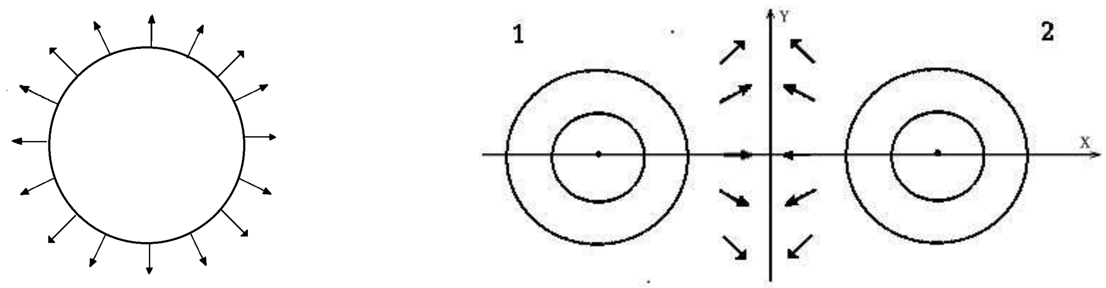

Figure 2: **Left:** isolated defect solution, $D = 2$, S^1-valued field. **Right:** problems with matching two $D = 2$ defects.

consistent with asymptotic behavior of *both* defects in this vicinity. To circumvent the second problem one can either insert "junctions" in between defects or consider multidefects configurations. Both ways seems to be not quite satisfactory.

In the case of TSs one doesn't encounter such difficulties. That is why we are interested in search of solitonic analogs of topological defects both in $D = 2$ and $D = 3$ cases; in other words, it is interesting and important to find soliton analogs of Nielsen-Olesen strings-vortices in $D = 2$ and of 't Hooft-Polyakov monopoles-hedgehogs in $D = 3$.

Topological Solitons in the A3M model. Instead of complex scalar field in the Abelian Higgs model (AHM) in the A3M model (see [1] and Refs. therein) we introduce 3-component unit isovector scalar field $s_a(\mathbf{x})$ taking values on unit sphere $S^2: \ s_a s_a = 1, \ a = 1, 2, 3$, having selfinteraction of so-called "easy-axis" type (well-known in the magnetism theory). Similar to AHM introduce gauge-invariant interaction of this field with the Maxwell field, making global $U(1)$ symmetry of easy-axis magnets local one. As a results we arrive at "the $A3M$ model". The Lagrangian density of the A3M-model is

$$\begin{aligned} \mathcal{L} \ &= \bar{\mathcal{D}}_\mu s_- \mathcal{D}^\mu s_+ + \partial_\mu s_3 \partial^\mu s_3 - V(s_a) - \frac{1}{4} F^2_{\mu\nu}, \\ &\bar{\mathcal{D}}_\mu = \partial_\mu + ieA_\mu, \quad \mathcal{D}_\mu = \partial_\mu - ieA_\mu, \\ &s_+ = s_1 + is_2, \quad s_- = s_1 - is_2, \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad V(s_a) = \beta(1 - s_3^2), \end{aligned} \quad (1)$$

$\mu, \nu = 0, 1, ..., D$. This model is the gauge-invariant extension of the classical Heisenberg antiferromagnet model with the easy-axis anisotropy. It supports $D = 2$ topological solitons, which can be found using the "hedgehog" ansatz for the unit isovector field $s_i(\mathbf{x})$, $i = 1, 2, 3$,

$$\begin{aligned} s_1 \ &= \cos m\chi \sin \vartheta(R), \quad s_2 = \sin m\chi \sin \vartheta(R), \\ &s_3 = \cos \vartheta(R), \\ &\sin \chi = \frac{y}{R}, \quad \cos \chi = \frac{x}{R}, \quad R^2 = x^2 + y^2, \end{aligned} \quad (2)$$

where m is an integer number, and the "vortex" ansatz for the Maxwell field $A_\mu(\mathbf{x})$,

$$\begin{aligned} A_0 = \ 0, \quad A_1 = A_x = -ma(R)\frac{y}{R^2}, \\ A_2 = A_y = ma(R)\frac{x}{R^2}. \end{aligned} \quad (3)$$

The topological charge of $A3M$ solitons is defined as the mapping degree of $s_a(\mathbf{x})$ distributions inside infinite radius ($R = \infty$) sphere, $R^2_{comp} \to S^2$. Boundary conditions correspond to uniform distribution of the $s^a(x)$ field at $R = \infty$, and zero value of the Maxwell field $A_\mu(x)$ at space infinity. $A3M$ solitons exist for integer Q_{top} – similar to Belavin-Polyakov $2D$ solitons in *isotropic* Heisenberg magnet.

Energy of two solitons with $Q_{top} = 1$ proves to be greater than energy of one soliton with $Q_{top} = 2$. As a result two such solitons attract to each other and coalesce into one $Q_{top} = 2$ soliton. Thus soliton analogs of Nielsen-Olesen TDs in the AHM have been found.

Topological Solitons in the SU2-Higgs model. Consider the simplest electroweak (EW) model (a reduction of the bosonic sector of the Weinberg-Salam model), the so-called SU2-Higgs model with

$$\begin{aligned} \mathcal{L} \ &= (\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - (\Phi^\dagger \Phi - 1)^2, \\ &\mathcal{D}_\mu \Phi_b = \partial_\mu \Phi_b + \frac{i}{2} g \tau^a A^a_\mu \Phi_b, \\ &\mu = 0, 1, 2, 3, \quad a = 1, 2, 3, \quad b = 1, 2, \end{aligned} \quad (4)$$

here Φ is the 2-component complex isospinor, defined by 4 real numbers φ_c, such that $\varphi_c \varphi_c = 1, c = 1, 2, 3, 4$. Introduce unit isospinor field

$$\tilde{\Phi} = \Phi / \sqrt{\Phi^\dagger \Phi} \quad \text{and} \quad \tilde{\varphi}_a = \varphi_a / \sqrt{\varphi_c \varphi_c}, \quad (5)$$

$a = 1, 2, 3, 4$ so that normalized field $\tilde{\varphi}_a$ takes values on unit sphere S^3. The SU2-Higgs model describes gauge-invariant interaction of $SU(2)$ Yang-Mills field with the isospinor scalar field. Let boundary conditions at $R = \infty, R^2 = x^2 + y^2 + z^2$: be $\tilde{\varphi}^c(\infty) = \tilde{\varphi}^c_0$, with $\tilde{\varphi}^c_0 = (0, 0, 0, 1)$, or $\tilde{\varphi}^c_0 = (0, 0, 0, -1)$. The topological charge Q_{top} can be defined as the mapping degree of $R^3_{comp} \to S^3$ given by distribution of the 4-component unit field $\tilde{\phi}_a(x)$ inside infinite radius sphere $R = \infty$.

Existence of topological solitons with integer topological charge Q_{top} is not *a priori* excluded. To find TSs one can use (i) hedgehog ansatz for isospinor field with chosen Q_{top}; in the simplest case $Q_{top} = 1$ it takes the form

$$\begin{gathered}\tilde{\phi}_4 = \cos\psi(r)\cdot f(r),\ \tilde{\phi}_3 = \sin\psi(r)\cdot\cos\theta(r)\cdot f(r),\\ \tilde{\phi}_2 = \sin\psi(r)\cdot\sin\theta(r)\cdot\sin\phi(x,y)\cdot f(r),\\ \tilde{\phi}_1 = \sin\psi(r)\cdot\sin\theta(r)\cdot\cos\phi(x,y)\cdot f(r),\\ \sin\phi = y/\sqrt{(x^2+y^2)}, \cos\phi = x/\sqrt{(x^2+y^2)}, \quad (6)\end{gathered}$$

here $\phi(r)$, $\theta(r)$ and $f(r)$ are to be found by minimization of topological lump energy and (ii) generic 3-term ansatz for $D = 3$ Yang-Mills solitons ($A^a_0 = 0$):

$$\begin{aligned} gA^a_i &= \varepsilon_{iak}\frac{x_k}{R^2}s(R)\\ &+ \frac{b(R)}{R^3}\left[(\delta_{ia}R^2 - x_i x_a) + \frac{p(R)x_i x_a}{R^4}\right],\\ i,k &= 1,2,3 \qquad R^2 = x^2+y^2+z^2. \end{aligned} \quad (7)$$

Study of Topological Solitons in the SU(2)-Higgs model is in progress.

Note that the SU(2)-Higgs model does not support TDs, because the 4-component unit field defined on the sphere S^2 has no nontrivial topological properties. This however does not mean that there is no possibility for existence of TSs in this model because maps ${R^3}_{comp} \to S^3$ are divided into classes with different integer topological charges. This example shows that usage of the generic term soliton for TDs and TSs can lead to erroneous conclusion.

In paper [9] an attempt was undertaken to study nonperturbative properties of models comprising the Yang-Mills fields in Coulomb gauge. New ansatz was proposed for classical description of extended string-like configurations in the $D = 2$ Coulomb gauge $SU(2)$ gluodynamics which can be used when studying interquark gluonic strings in mesons. Applying Coulomb gauge condition and hedgehog ansatzes allowed to separate angular and radial variables and obtain expressions for Hamiltonian densities for $D = 2, 3$ pure $SU(2)$ Yang-Mills models and $D = 3$ $SU(2)$-Higgs gauged nonlinear σ-model.

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