

Renormdynamics and Hamiltonization of Dynamical Systems

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Renormdynamics

Quantum field theory (QFT) and Fractal calculus (FC) provide Universal language of fundamental physics (see e.g. [9]). In QFT existence of a given theory means, that we can control its behavior at some scales (short or large distances) by renormalization theory [3]. If the theory exists, than we want to solve it, which means to determine what happens on other (large or short) scales. This is the problem (and content) of Renormdynamics. The result of the Renormdynamics, the solution of its discrete or continual motion equations, is the effective QFT on a given scale (different from the initial one).

Perturbation theory series (PTS) have the following qualitative form

$$\begin{aligned} f(x) &= \sum_{n \geq 0} P(n)n!x^n \\ &= P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \quad \delta = x\frac{d}{dx} \end{aligned} \quad (1)$$

So, we reduce previous series to the standard geometric progression series. This series is convergent for $|x| < 1$ or for $|x|_p = p^{-k} < 1$, $x = p^k a/b$, $k \geq 1$, $p = 2, 3, 5, \dots, 29, \dots, 137, \dots$. With an appropriate normalization of the expansion parameter, the coefficients of the series are rational numbers and if experimental data indicates for some prime value for x , e.g. in QED, $x = \alpha = e^2/(4\pi) = 1/137.036\dots$, then we can take corresponding prime number and consider p-adic convergence of the series.

In the Yukawa theory of strong interactions (see e.g. [1]), we take $x = \alpha_{\pi N} = 13$,

$$|f|_p \leq \sum |f_n|_p p^{-n} < \frac{1}{1-p^{-1}}, \quad p = 13 \quad (2)$$

So, the series is convergent. If the limit is rational number, we consider it as an observable value of the corresponding physical quantity.

In *MSSM* (see [6]) coupling constants unifies at $\alpha_u^{-1} = 26.3 \pm 1.9 \pm 1$. So, $23.4 < \alpha_u^{-1} < 29.2$

Question: how many primes are in this interval?

$$24, 25, 26, 27, 28, 29 \quad (3)$$

Only one!

Proposal: take the value $\alpha_u^{-1} = 29.0\dots$ which will be two orders of magnitude more precise prediction and find the consequences for the *SM* scale observables.

Goldberger-Treiman relation and the pion-nucleon coupling constant

The Goldberger-Treiman relation (GTR) [5] plays an important role in theoretical hadronic and nuclear physics. GTR relates the Meson-Nucleon coupling constants to the axial-vector coupling constant in β -decay: $g_{\pi N} f_\pi = g_A m_N$, where m_N is the nucleon mass, g_A is the axial-vector coupling constant in nucleon β -decay at vanishing momentum transfer, f_π is the π decay constant and $g_{\pi N}$ is the π - N coupling constant.

Since the days when the Goldberger-Treiman relation was discovered, the value of g_A has increased considerably. Also, f_π decreased a little, on account of radiative corrections. The main source of uncertainty is $g_{\pi N}$. When we take

$$\alpha_{\pi N} = \frac{g_{\pi N}^2}{4\pi} = 13 \Rightarrow g_{\pi N} = 12.78, \quad (4)$$

experimental value for f_π from pion decay and neutron mass

$$f_\pi = \frac{130}{\sqrt{2}} = 91.9 \text{ MeV}, \quad m_N = 940 \text{ MeV}, \quad (5)$$

we find

$$\begin{aligned} g_A &= \frac{f_\pi g_{\pi N}}{m_N} = \frac{91.9 \times \sqrt{52}\pi}{940} \\ &= 1.2496 \simeq 1.25 = \frac{5}{4} \end{aligned} \quad (6)$$

Renormdynamics of QCD

QCD is the theory of the strong interactions with, as only inputs, one mass parameter for each quark species and the value of the QCD coupling constant at some energy or momentum scale in some renormalization scheme. This last free parameter of the theory can be fixed by Λ_{QCD} , the energy scale used as the typical boundary condition for the integration of the Renormdynamic (RD) equation for the strong coupling constant. This is the parameter which expresses the scale of strong interactions, the only parameter in the limit of massless quarks. While the evolution of the coupling with the momentum scale is determined by the quantum corrections induced by the renormalization of the bare coupling and can be computed in perturbation theory, the strength itself of the interaction, given at

any scale by the value of the renormalized coupling at this scale, or equivalently by Λ_{QCD} , is one of the above mentioned parameters of the theory and has to be taken from experiment.

The RD equation for the coupling constant is

$$\dot{a} = \beta(a) = \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4 + \beta_5 a^5 + \dots$$

$$\int_{a_0}^a \frac{da}{\beta(a)} = t - t_0 = \ln \frac{\mu^2}{\mu_0^2}, \quad (7)$$

μ is the 't Hooft unit of mass, the renormalization point in the MS-scheme. To calculate the β -function we need to calculate the renormalization constant Z of the coupling constant, $a_b = Za$, where a_b is the bare (unrenormalized) charge. The expression of the β -function can be obtained in the following way

$$0 = d(a_b \mu^{2\varepsilon})/dt = \mu^{2\varepsilon} (\varepsilon Z a + \frac{\partial(Za)}{\partial a} \frac{da}{dt})$$

$$\Rightarrow \frac{da}{dt} = \beta(a, \varepsilon) = \frac{-\varepsilon Z a}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),$$

$$\beta(a) = a \frac{d}{da} (a Z_1), \quad (8)$$

Z_1 is the residue of the first pole in ε expansion

$$Z(a, \varepsilon) = 1 + Z_1 \varepsilon^{-1} + \dots + Z_n \varepsilon^{-n} + \dots \quad (9)$$

Reparametrization and general method of solution of the RD equation

RD equation,

$$\dot{a} = \beta_1 a + \beta_2 a^2 + \dots \quad (10)$$

can be reparametrized,

$$a(t) = f(A(t)) = A + f_2 A^2 + \dots$$

$$+ f_n A^n + \dots = \sum_{n \geq 1} f_n A^n, \quad (11)$$

$$\dot{A} = b_1 A + b_2 A^2 + \dots = \sum_{n \geq 1} b_n A^n,$$

$$\dot{a} = \dot{A} f'(A) =$$

$$= \beta_1 A + (\beta_2 + \beta_1 f_2) A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3) A^3 +$$

$$\dots + (\beta_n + (n-1)\beta_{n-1} f_2 + \dots + \beta_1 f_n) A^n + \dots$$

$$= \sum_{n, n_1, n_2 \geq 1} A^n b_{n_1} n_2 f_{n_2} \delta_{n, n_1 + n_2 - 1},$$

$$b_1 = \beta_1, \quad b_2 = \beta_2 + f_2 \beta_1 - 2f_2 b_1 = \beta_2 - f_2 \beta_1,$$

$$b_3 = \beta_3 + 2f_2 \beta_2 + f_3 \beta_1 - 2f_2 b_2 - 3f_3 b_1$$

$$= \beta_3 + 2(f_2^2 - f_3) \beta_1,$$

$$b_4 = \beta_4 + 3f_2 \beta_3 + f_2^2 \beta_2$$

$$+ 2f_3 \beta_2 - 3f_4 b_1 - 3f_3 b_2 - 2f_2 b_3, \dots$$

$$b_n = \beta_n + \dots + \beta_1 f_n$$

$$- 2f_2 b_{n-1} - \dots - n f_n b_1, \dots \quad (12)$$

so, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1 .

We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{\beta_2}{\beta_1}, \quad f_3 = \frac{\beta_3}{2\beta_1} + f_2^2, \quad \dots, \quad f_n = \frac{\beta_n + \dots}{(n-1)\beta_1}, \dots \quad (13)$$

In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but β_2 and β_3 .

From the relations (12), in the critical dimension ($\beta_1 = 0$), we find that, we can define the minimal form of the RD equation

$$\dot{A} = \beta_2 A^2 + \beta_3 A^3, \quad (14)$$

We can solve (14) as implicit function,

$$u^{\beta_3/\beta_2} e^{-u} = c e^{\beta_2 t}, \quad u = \frac{1}{A} + \frac{\beta_3}{\beta_2} \quad (15)$$

then, as in the noncritical case, explicit solution for a will be given by reparametrization representation (11) [10].

If we know somehow the coefficients β_n , e.g. for first several exact and for others asymptotic values (see e.g. [7]) than we can construct reparametrization function (11) and find the dynamics of the running coupling constant. This is similar to the action-angular canonical transformation of the analytic mechanics (see e.g. [4]).

Statement: The reparametrization series for a is p -adically convergent, when β_n and A are rational numbers.

QCD, parton model, valence quarks and $\alpha_s = 2$

While it has been well established in the perturbative regime at high energies, QCD still lacks a comprehensive solution at low and intermediate energies, even 40 years after its invention. In order to deal with the wealth of non-perturbative phenomena, various approaches are followed with limited validity and applicability. This is especially also true for lattice QCD, various functional methods, or chiral perturbation theory, to name only a few. In neither one of these approaches the full dynamical content of QCD can yet be included. Basically, the difficulties are associated with a relativistically covariant treatment of confinement and the spontaneous breaking of chiral symmetry, the latter being a well-established property of QCD at low and intermediate energies. As a result, most hadron reactions, like resonance excitations, strong and electroweak decays etc., are nowadays only amenable to models of QCD. Most famous is the constituent-quark model (CQM), which essentially relies on a limited number of effective degrees of freedom with

the aim of encoding the essential features of low- and intermediate-energy QCD.

The CQM has a long history, and it has made important contributions to the understanding of many hadron properties, think only of the fact that the systematization of hadrons in the standard particle-data base follows the valence-quark picture. The Q dependence of the nucleon form factor corresponds to three-constituent picture of the nucleon and is well described by the simple expression [2], [8]

$$F(Q^2) \sim (Q^2)^{-2} \quad (16)$$

It was noted [12] that parton densities given by the following solution

$$\begin{aligned} M_2(Q^2) &= \frac{3}{25} + \frac{2}{3}\omega^{32/81} + \frac{16}{75}\omega^{50/81}, \\ \bar{M}_2(Q^2) &= M_2^s(Q^2) = \frac{3}{25} - \frac{1}{3}\omega^{32/81} + \frac{16}{75}\omega^{50/81}, \\ M_2^G(Q^2) &= \frac{16}{25}(1 - \omega^{50/81}), \\ \omega &= \frac{\alpha_s(Q^2)}{\alpha_s(m^2)}, \quad Q^2 \in (5, 20) \text{ GeV}^2, \\ b &= 11 - \frac{2}{3}n_f = 9, \quad \alpha_s(Q^2) \simeq 0.2 \end{aligned} \quad (17)$$

of the Altarelli-Parisi equation

$$\begin{aligned} \dot{M} &= AM, \\ M^T &= (M_2, \bar{M}_2, M_2^s, M_2^G), \\ M_2 &= \int_0^1 dx x(u(x) + d(x)), \\ \bar{M}_2 &= \int_0^1 dx x(\bar{u}(x) + \bar{d}(x)), \\ M_2^s &= \int_0^1 dx x(s(x) + \bar{s}(x)), \quad M_2^G = \int_0^1 dx x G(x), \\ A &= -a(Q^2) \begin{pmatrix} 32/9 & 0 & 0 & -2/3 \\ 0 & 32/9 & 0 & -2/3 \\ 0 & 0 & 32/9 & -2/3 \\ -32/9 & -32/9 & -32/9 & 2 \end{pmatrix} \\ a &= \left(\frac{g}{4\pi}\right)^2, \quad \dot{M} = Q^2 \frac{dM}{dQ^2} \end{aligned} \quad (18)$$

with the following valence quark initial condition at a scale m

$$\begin{aligned} \bar{M}_2(m^2) &= M_2^s(m^2) = M_2^G(m^2) = 0, \\ M_2(m^2) &= 1 \end{aligned} \quad (19)$$

and

$$\alpha_s(m^2) = 2, \quad (20)$$

gives the experimental values

$$M_2 = 0.44, \quad \bar{M}_2 = M_2^s = 0.04, \quad M_2^G = 0.48 \quad (21)$$

So, for valence quark model, $\alpha_s(m^2) = 2$. We have seen, that for $\pi\rho N$ model $\alpha_{\pi\rho N} = 3$, and for πN model $\alpha_{\pi N} = 13$. It is nice that $\alpha_s^2 + \alpha_{\pi\rho N}^2 = \alpha_{\pi N}$; to $\alpha_s = 2$ corresponds

$$g = \sqrt{4\pi\alpha_s} = 5.013 = 5+ \quad (22)$$

Hamiltonization of dynamical systems

Let us consider a general dynamical system described by the following system of the ordinary differential equations [13]

$$\dot{x}_n = v_n(x), \quad 1 \leq n \leq N, \quad (23)$$

\dot{x}_n stands for the total derivative with respect to the parameter t . When the number of the degrees of freedom is even, and

$$v_n(x) = \varepsilon_{nm} \frac{\partial H}{\partial x_m}, \quad 1 \leq n, m \leq 2M, \quad (24)$$

the system (23) is Hamiltonian one and can be put in the form

$$\dot{x}_n = \{x_n, H\}, \quad (25)$$

where the Poisson bracket is defined as

$$\{A, B\} = \varepsilon_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = A \overleftarrow{\frac{\partial}{\partial x_n}} \varepsilon_{nm} \overrightarrow{\frac{\partial}{\partial x_m}} B, \quad (26)$$

and summation rule under repeated indices has been used.

Let us consider the following Lagrangian

$$L = (\dot{x}_n - v_n(x))\psi_n \quad (27)$$

and the corresponding motion equations

$$\dot{x}_n = v_n(x), \quad \dot{\psi}_n = -\frac{\partial v_m}{\partial x_n} \psi_m. \quad (28)$$

The system (28) extends the general system (23) by linear equation for the variables ψ . The extended system can be put in the Hamiltonian form [20]

$$\dot{x}_n = \{x_n, H\}, \quad \dot{\psi}_n = \{\psi_n, H\}, \quad (29)$$

where Hamiltonian is

$$H = v_n(x)\psi_n \quad (30)$$

and the bracket is defined as

$$\{A, B\} = A \left(\overleftarrow{\frac{\partial}{\partial x_n}} \overrightarrow{\frac{\partial}{\partial \psi_n}} - \overleftarrow{\frac{\partial}{\partial \psi_n}} \overrightarrow{\frac{\partial}{\partial x_n}} \right) B. \quad (31)$$

In the Faddeev-Jackiw formalism [18] for the Hamiltonian treatment of systems defined by first-order Lagrangians,

$$L = f_n(x)\dot{x}_n - H(x), \quad (32)$$

motion equations

$$f_{mn}\dot{x}_n = \frac{\partial H}{\partial x_m}, \quad (33)$$

for the regular structure function f_{mn} , can be put in the explicit Hamiltonian (Poisson; Dirac) form

$$\dot{x}_n = f_{nm}^{-1} \frac{\partial H}{\partial x_m} = \{x_n, x_m\} \frac{\partial H}{\partial x_m} = \{x_n, H\}, \quad (34)$$

where the fundamental Poisson (Dirac) bracket is

$$\{x_n, x_m\} = f_{nm}^{-1}, \quad f_{mn} = \partial_m f_n - \partial_n f_m. \quad (35)$$

The system (28) is an important example of the first order regular Hamiltonian systems.

Indeed, in the new variables,

$$y_n^1 = x_n, y_n^2 = \psi_n, \quad (36)$$

Lagrangian (27) takes the following first order form

$$\begin{aligned} L &= (\dot{x}_n - v_n(x))\psi_n \\ &\Rightarrow \frac{1}{2}(\dot{x}_n\psi_n - \dot{\psi}_n x_n) - v_n(x)\psi_n \\ &= \frac{1}{2}y_n^a \varepsilon^{ab} \dot{y}_n^b - H(y) \\ &= f_n^a(y) \dot{y}_n^a - H(y), \\ f_n^a &= \frac{1}{2}y_n^b \varepsilon^{ba}, \quad H = v_n(y^1) y_n^2, \\ f_{nm}^{ab} &= \frac{\partial f_m^b}{\partial y_n^a} - \frac{\partial f_n^a}{\partial y_m^b} = \varepsilon^{ab} \delta_{nm}; \end{aligned} \quad (37)$$

corresponding motion equations and the fundamental Poisson bracket are

$$\begin{aligned} \dot{y}_n^a &= \varepsilon_{ab} \delta_{nm} \frac{\partial H}{\partial y_m^b} = \{y_n^a, H\}, \\ \{y_n^a, y_m^b\} &= \varepsilon_{ab} \delta_{nm}. \end{aligned} \quad (38)$$

The Hamiltonian mechanics (HM) is in the fundamentals of mathematical description of the physical theories [4]. But HM is in a sense blind; e.g., it does not make a difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) [29] and (super)integrable Hamiltonian systems (with maximal number of the integrals of motion). Nabu mechanics (NM) [28, 31] is a proper generalization of the HM, which makes the difference between dynamical systems with different numbers of integrals of motion explicit (see, e.g. [24]).

In the canonical formulation, the equations of motion of a physical system are defined via a Poisson bracket and a Hamiltonian, [13]. In Nambu formulation, the Poisson bracket is replaced by the Nambu bracket with $n+1, n \geq 1$, slots. For $n=1$, we have the canonical formalism with one Hamiltonian. For $n \geq 2$, we have Nambu-Poisson formalism, with n Hamiltonians, [28], [31].

The quasi-classical description of the motion of a relativistic point particle with spin in accelerators and storage rings includes the equations of orbit motion

$$\dot{x}_n = f_n(x), \quad f_n(x) = \varepsilon_{nm} \partial_m H, \quad n, m = 1, 2, \dots, 6;$$

$$\begin{aligned} x_n &= q_n, \quad x_{n+3} = p_n, \quad \varepsilon_{n,n+3} = 1, \quad n = 1, 2, 3; \\ H &= e\Phi + c\sqrt{\wp^2 + m^2 c^2}, \quad \wp_n = p_n - \frac{e}{c} A_n \end{aligned} \quad (39)$$

and Thomas-BMT equations [30, 14] of classical spin motion

$$\begin{aligned} \dot{s}_n &= \varepsilon_{nmk} \Omega_m s_k = \{H_1, H_2, s_n\}, \\ H_1 &= \Omega \cdot s, \quad H_2 = s^2, \\ \{A, B, C\} &= \varepsilon_{nmk} \partial_n A \partial_m B \partial_k C, \\ \Omega_n &= \frac{-e}{m\gamma c} ((1+k\gamma)B_n - k \frac{(B \cdot \wp) \wp_n}{m^2 c^2 (1+\gamma)} \\ &+ \frac{1+k(1+\gamma)}{mc(1+\gamma)} \varepsilon_{nmk} E_m \wp_k) \end{aligned} \quad (40)$$

where, parameters e and m are the charge and the rest mass of the particle, c is the velocity of light, $k = (g-2)/2$ quantifies the anomalous spin g factor, γ is the Lorentz factor, p_n are components of the kinetic momentum vector, E_n and B_n are the electric and magnetic fields, and A_n and Φ are the vector and scalar potentials;

$$\begin{aligned} B_n &= \varepsilon_{nmk} \partial_m A_k, \quad E_n = -\partial_n \Phi - \frac{1}{c} \dot{A}_n, \\ \gamma &= \frac{H - e\Phi}{mc^2} = \sqrt{1 + \frac{\wp^2}{m^2 c^2}} \end{aligned} \quad (41)$$

The spin motion equations we put in the Nambu-Poisson form. Hamiltonization of this dynamical system according to the general approach of the previous sections we will put in the ground of the optimal control theory of the accelerator.

The general method of Hamiltonization of the dynamical systems we can use also in the spinning particle case. Let us invent unified configuration space $q = (x, p, s)$, $x_n = q_n$, $p_n = q_{n+3}$, $s_n = q_{n+6}$, $n = 1, 2, 3$; extended phase space, (q_n, ψ_n) and Hamiltonian

$$H = H(q, \psi) = v_n \psi_n, \quad n = 1, 2, \dots, 9; \quad (42)$$

motion equations

$$\dot{q}_n = v_n(q), \quad \dot{\psi}_n = -\frac{\partial v_m}{\partial q_n} \psi_m \quad (43)$$

where v_n depends on external fields as in previous section as control parameters which can be determined according to the optimal control criterium.

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