Klein-Gordon Equations in the Rapidity Space and Generalized de Broglie Formula

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Introduction

The mass-shell equation is one of the basic relationships of the relativistic mechanics. This equation is given by the quadratic polynomial

$$p_0^2 = p^2 + m^2c^2, \tag{1.1}$$

where $p^2 = p_x^2 + p_y^2 + p_z^2$, m is the proper mass and p_0 means a non-potential part of the total energy E: $E = p_0 c + V(r)$. It easily seen that the mass-shell equation can be interpreted as Pythagoras theorem of right-angled triangle. This triangle has one independent angle which can be done by periodic or hyperbolic angle. In the relativistic kinematics this parameter is denominated as the *rapidity*.

In this paper we review the articles [1], [2], [3], [4], [5], [6], [7] where we have introduced concept of a *counterpart of rapidity* (co-rapidity) and demonstrated its usefulness for relativistic dynamics. The dynamic equations with respect to co-rapidity leads to solution of Riccati equation. There also it has been established a relationship between four-rapidity and space-time coordinates. We have shown that the rapidity can be presented as a four-vector with time-like and space-like coordinates. In this space-time a relativistic motion is described by an analogue of Klein-Gordon equation.

Key-formulae linking an exponential function with ratio of two quantities

Our construction is based on the Key-formula which establishes some natural interrelation between the ratio of a pair of quantities and an exponential function.

2.1 Parametrization of evolution with respect to hyperbolic angle.

Let p_0, p be components of the energy-momentum of relativistic particle. Then, with respect to the co-rapidity an evolution of the relativistic particle is generated by the following quadratic polynomial

$$F(X) := X^2 - 2p_0X + p^2, \tag{2.1}$$

with distinct positive real roots x_1, x_2, so that,

$$2p_0 = x_1 + x_2, \quad p^2 = x_1x_2. \tag{2.2}$$

The coefficients p_0, p^2 are real numbers and $p_0^2 > p^2$. The solutions of equation (2.1) are defined by

$$x_1 = p_0 + mc, \quad x_2 = p_0 - mc, \quad mc = +\sqrt{p_0^2 - p^2}, \tag{2.3}$$

where m is the proper-mass. The accompanying matrix of the polynomial $F(X)$ is defined by

$$E = \begin{pmatrix} 0 & -p^2 \\ 1 & 2p_0 \end{pmatrix}. \tag{2.4}$$

Consider an evolution generated by matrix E. Write the Euler formula

$$\exp(E\phi) = E\ g_1(\phi) + I\ g_0(\phi), \tag{2.5}$$

I-is a unit matrix. Form the following ratio

$$\exp((x_2 - x_1)\phi) = \frac{x_2\ g_1(\phi) + g_0(\phi)}{x_1\ g_1(\phi) + g_0(\phi)} = \frac{x_2 + D}{x_1 + D}, \tag{2.6}$$

where

$$D = \frac{g_0(\phi)}{g_1(\phi)}. \tag{2.7}$$

Let $\phi = \phi_0$ be the point where $g_0(\phi_0) = 0$. Then,

$$\exp((x_2 - x_1)\phi_0) = \frac{p_0 + m}{p_0 - m}. \tag{2.8}$$

Consequently, we have the following dependence p_0, p of ϕ_0:

$$p_0(\phi_0) = m\coth(m\phi_0), \quad p(\phi_0) = \frac{m}{\sinh(m\phi_0)}. \tag{2.9}$$

The Key-formula (2.6) closely related with definition of the cross-ratio. In fact, from (2.7) it follows

$$\exp((x_2 - x_1)(\phi_u - \phi_v)) = \frac{x_2 - D(\phi_u)}{x_1 - D(\phi_u)}\frac{x_1 - D(\phi_v)}{x_2 - D(\phi_v)}. \tag{2.10}$$

Hence, the quantity under exponential function, i.e. the logarithm of the cross-ratio,

$$(x_2 - x_1)(\phi_u - \phi_v) = \log\{\frac{x_2 - D(\phi_u)}{x_1 - D(\phi_u)}\frac{x_1 - D(\phi_v)}{x_2 - D(\phi_v)}\} \tag{2.11}$$

is the distance in half-plane model of Lobachevsky space proposed by Poincaré.

Now, consider the quadratic polynomial

$$F(Y) := Y^2 - 2pY + p_0^2, \tag{2.12}$$

which differs from (2.1) by transposition of the coefficients p_0 and p. Since $p_0^2 > p^2$, two solutions of equation $F(Y) = 0$ are given by complex conjugated numbers:

$$y_2 = p+im, \;\; y_1 = p-im, \;\; m = +\sqrt{p_0^2 - p^2}. \quad (2.13)$$

In these terms the Key formula is written as

$$\exp(i2m\theta) = \frac{p(\theta) + im}{p(\theta) - im}. \quad (2.14)$$

From this formula it follows another representation of the energy-momentum

$$p(\theta) = m\cot(m\theta), \quad p_0(\theta) = m\frac{1}{\sin(m\theta)}. \quad (2.15)$$

Since formulae (2.9) and (2.15) are related to same physical quantities, we come to the next relationships between hyperbolic and periodic trigonometric functions

$$\tanh(m\phi) = \sin(m\theta), \text{ or, } \sinh(m\phi) = \tan(m\theta). \quad (2.16)$$

The relationships between ϕ and θ can be presented also as follows

$$\exp(m\phi) = \frac{1+\sin(m\theta)}{1-\sin(m\theta)} = \frac{1+\tan\frac{m\theta}{2}}{1-\tan\frac{m\theta}{2}}, \quad (2.17a)$$

$$\exp(im\theta) = \frac{1+i\sinh(m\phi)}{1-i\sinh(m\phi)} = \frac{1+i\tanh\frac{m\phi}{2}}{1-i\tanh\frac{m\phi}{2}}, \quad (2.17b)$$

or in more compact form

$$\tan\frac{m\theta}{2} = \tanh\frac{m\phi}{2}. \quad (2.18)$$

Notice, when $m = 0$, $\phi = \theta$.

Also, it is important to note that the differentiation θ with respect to ϕ coincides with the differentiation of the distance with respect to coordinate time, i.e. with the velocity:

$$\frac{d\theta}{d\phi} = \frac{dr}{dt} = \frac{p}{p_0}. \quad (2.19)$$

Klein-Gordon equations for energy-momentum of classical relativistic particle in the space of rapidity

For the sake of convenience in this section let us use for derivatives short notations

$$\frac{d}{d\phi} = d, \quad \frac{d}{d\theta} = \partial.$$

Then differentiating formulae (2.9) and (2.15) we come to the following system of differential equations

$$dp_0 = -p^2, \quad dp = -pp_0, \quad \partial p_0 = -pp_0, \quad \partial p = -p_0^2.$$

The operators d and ∂ do not commute. Introduce two dimensional vector of a state by

$$\Phi(p_0, p) = \begin{pmatrix} p_0 \\ p \end{pmatrix}.$$

Calculate actions of the operators $d^2 - \partial^2$ and $d\partial - \partial d$ on this vector:

$$(d^2 - \partial^2)\Phi(p_0, p) = m^2 \; \Phi(p_0, p),$$

$$(\partial d - d\partial)\Phi(p_0, p) = m^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi(p_0, p). \quad (3.1)$$

It is seen that equation (3.1) is nothing else than two dimensional Klein-Gordon equation. Comparing this equation with two dimensional Klein-Gordon equation written in terms of space-time coordinates, also taking into account (2.18), we come to conclusion that the parameter ϕ is a time-like parameter, whereas the parameter θ is an analogue of a space coordinate. In order to pass to the Klein -Gordon equation in four-dimensional Minkowski space with signature $(+---)$ we shall extend the parameter θ till to three dimensional vector. In this way we arrive to covariant formulation of evolution equations.

The momentum is a spatial part of the four-vector energy-momentum with components $p_k, k = 1, 2, 3$. Now, instead of ϕ we will use the letter ρ_0, and θ has to be replaced by spatial part of four-vector of rapidity containing components ρ_1, ρ_2, ρ_3.

In these variables the evolution equations have to be written in a Lorentz-covariant form. The evolution equations we shall extend as follows. The single variable p is replaced by the components of three-vector of momentum, $p_k, k = 1, 2, 3$. The square p^2 means $p^2 = -p^k p_k$. In this way we arrive to the following set of equations

$$(a) \;\; d^0 \, p_0 = \; -p^k p_k, \;\; (b) \;\; d^0 \, p_k = p_k p^0, \;\; k = 1, 2, 3. \quad (3.2)$$

Hereafter we use the following notations for derivatives

$$\partial^k = \frac{\partial}{\partial \rho_k}, \quad d^0 = \frac{d}{d\rho_0},$$

and adopt, so-called, *a summation convention*, according to which any repeated index in one term, once up, once down, implies summation over all its values.

Remember, however, that there exist some functional dependence between ρ_0 and $\rho_k, k = 1, 2, 3$, so that the spatial variables are functions of the time-like parameter, i.e., $\rho_k = \rho_k(\rho_0), k = 1,2,3$. This means, the full derivative with respect to ρ_0 is

$$d^0 p_0 = -p^k p_k = \frac{d\rho_k}{d\rho_0}\frac{\partial}{\partial \rho_k} p_0. \qquad (3.3)$$

On making use of equations (3.2), we get

$$d^0 p_0 = p^2 = -p^k p_k = p_k \frac{d\rho_k}{d\rho_0} p_0. \qquad (3.4)$$

In order to provide this equality we have to take

$$p_k \frac{d\rho_k}{d\rho_0} = \frac{p^2}{p_0}. \qquad (3.5)$$

Our purpose is to complete the evolution equations (3.2) with an equation containing the following derivative

$$\frac{\partial}{\partial \rho_n} p_k.$$

For that reason let us re-write equation (3.1) as follows

$$\frac{d}{d\rho_0} p_k = \frac{d\rho_n}{d\rho_0}\frac{\partial}{\partial \rho_n} p_k = p_k p_0.$$

In order to provide this equality we have to suppose that

$$\frac{\partial}{\partial \rho_n} p_k = p_k p^n \frac{p_0^2}{p^2}. \qquad (3.6)$$

One may easily check that formula (3.6) is in accordance with (3.1) and (3.2).

Equations with second order derivatives.

Firstly, calculate the second order derivatives of p_0 and p with respect to time-like variable ρ_0. We have,

$$\frac{d}{d\rho^0}\frac{d}{d\rho_0} p_0 = -2p^k p_k p_0 = 2p^2 p_0. \qquad (3.7)$$

Secondly, calculate action of the Laplace operator on p_0:

$$\partial^k \partial_k \; p_0 = -p_0^2 p_0 + p^k p_k p_0 = -p_0^3 - p^2 p_0. \qquad (3.8)$$

Joining this equation with (3.7) we come to Klein-Gordon equation for p_0

$$d^0 d_0 \; p_0 + \partial^k \partial_k \; p_0 = -m^2 \; p_0. \qquad (3.9)$$

Now calculate action the Laplace operator on p_k. On making use of formulae (3.6) and (3.2) we get

$$\partial^n \partial_n \; p_k = \partial^n \; (\; p_k p_n \frac{p_0^2}{p^2} \;) = -2 p_k p_0^2. \qquad (3.10)$$

Joining this result with

$$d^0 d_0 \; p_k = p_k p_0^2 - p_k p^n p_n = p_k p_0^2 + p_k p^2,$$

we come to analogue of Klein-Gordon equation for p_k:

$$\Delta \; p_k + d^0 d_0 \; p_k \; = -m^2 \; p_k. \qquad (3.11)$$

From formula (3.5) it follows

$$\frac{d\rho^k}{d\rho_0} = v^k + M^{kl} p_l, \qquad (3.12)$$

where

$$v^k = \frac{dx^k}{dx^0}$$

is the velocity with respect to coordinate time, $M^{kl} = -M^{lk}$ is an arbitrary anti-symmetric tensor.

In the relativistic quantum mechanics the Klein-Gordon equation is obtained simply by using some conventional receipt according to which components of four-momentum in the mass-shell equation are replaced by corresponding differential operators as follows

$$p_k = -i\hbar \frac{\partial}{\partial x^k}, \; p_0 = i\hbar \frac{\partial}{\partial x^0}.$$

So, we come to the following correspondence

$$\hbar \rho^\mu \Rightarrow x^\mu, \;\; \frac{\partial}{\partial \rho^\mu} = \hbar \frac{\partial}{\partial x^\mu}. \qquad (3.13)$$

Extension of de Broglie formulae

Define energy-momentum with respect to variables inverse to ϕ, θ:

$$\phi = \frac{1}{\rho_0}, \quad \theta = \frac{1}{\rho}. \qquad (4.1)$$

In these terms,

$$p_0 = \coth(\frac{mc}{\rho_0}), \quad p = \cot(\frac{mc}{\rho}). \qquad (4.2)$$

For the small small values of proper mass

$$mc << \rho_0, \quad mc << \rho,$$

the following expansions hold true

$$p_0 = \frac{\rho_0}{mc} + ..., \quad p = \frac{\rho}{mc} + ...$$

Hence at the point $m = 0$ we get

$$\rho_0 = p_0(m=0), \quad rho = p(m=0)$$

The particle in the massless state displays properties of a wave. Energy of this particle is proportional to the frequency,

$$p_0 = \hbar\omega = h\nu = \rho_0,$$

and the momentum is proportional to wave number k and inverse to the length of the wave

$$p = \hbar k = \frac{h}{\lambda}, \qquad (4.3)$$

where h is Plank's constant. Notice, according to this hypothesis, $\lambda = \theta$ and $\phi = T$, where T is a period of the oscillation. Furthermore, now the relation

$$p = mv = \frac{h}{\lambda}, \quad (4.4)$$

can be considered as a first member of expansion of the equation

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} = mc\cot(\frac{mc}{\rho}), \quad (4.5)$$

for small values of $mc\rho$, and v^2/c^2. Here m is the proper mass. For small values of the proper mass we have

$$\frac{v}{\sqrt{1 - v^2/c^2}} = c\cot(\frac{m_0 c\lambda}{h}) = \frac{h}{m_0\lambda}. \quad (4.6)$$

or,

$$m_r v = \frac{h}{\lambda}, \ m_r = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$

So, we recovered de Broglie formula and realized that the mass m_r in this formula is the relativistic mass, depending of the velocity.

We think, the energy of the particle E can be equalized with the frequency by Plank's formula only if only the proper mass of the particle is zero. In order to apply the Plank-Einstein formula

$$E = p_c = h\nu, \quad p = \frac{h}{\lambda}, \quad (4.7)$$

which is true for the massless state, to the states with non-vanishing proper mass, we need some formula of mapping from the massless state onto the massive one. This mapping is given by formulae

$$E = p_0 c = mc\coth(\frac{mc}{\mu_0}), \ p = mc\cot(\frac{mc}{\mu}), \quad (4.8)$$

where μ_0, μ are the energy-momentum at the massless-state. At the massless state

$$\mu_0 c = h\nu, \quad \mu = \frac{h}{\lambda}.$$

Mapping these formulae for energy-momentum onto the state with non-zero proper mass we get

$$E = p_0 c = mc\coth(\frac{mc}{h\nu}), \ p = mc\cot(\frac{mc\lambda}{h}). \quad (4.9)$$

Relation λ with proper mass m and the velocity v is given by the following formula

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} = \frac{h}{\lambda} + mc\sum_n (\frac{1}{\frac{mc\lambda}{h} + n\pi} + \frac{1}{\frac{mc\lambda}{h} - n\pi}). \quad (4.10)$$

Remaining only the first term of the sum in right-hand side straightforward leads to the celebrated de Broglie relationship between momentum and the wave length. This approximation implies small value of the proper mass. The particles with small but finite value of proper masses can move with any velocity $v < c$. On the other hand, we can-not use infinitesimal value of the mass, because in this case the velocity of the particle has to tend to light-speed, which does not permit the formula on the left-hand side.

In the representation for momentum as a function of the time-like parameter we also can use a series:

$$p = 2mc\frac{1}{2\sinh(mc\phi)} = 2\sum_{n=0}^{\infty}(\ \exp(-2mc(n + \frac{1}{2})\phi). \quad (4.11)$$

$$2mc\sum_{n=0}^{\infty}(\ \exp(-2mc(n + \frac{1}{2})\phi) =$$

$$= \frac{1}{\chi} + mc \sum_{n=1}^{\infty}(\frac{1}{mc\chi + n\pi} + \frac{1}{mc\chi - n\pi}). \quad (4.12)$$

Joining these formulae into unique sum, we get

$$2mc\exp(-mc\phi) - \frac{1}{\chi} = 2\sum_{n=1}^{\infty}(-\exp(-(2mc(n + \frac{1}{2})\phi) +$$

$$+ mc \sum_{n=1}^{\infty}(\frac{mc}{mc\chi + n\pi} + \frac{mc}{mc\chi - n\pi}). \quad (4.13)$$

Now, remember the following formulae for sine- and cotangent functions

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2}),$$

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}(\frac{1}{z + n} + \frac{1}{z - n}). \quad (4.14)$$

Now, apply the first formula for the energy. Because,

$$P_0 = \frac{mc}{\sin(mc\theta)},$$

we write

$$P_0 = mc \ \frac{1}{\sin(\pi mc\theta\frac{1}{\pi})} = mc \ \frac{1}{\pi z}\prod_{n=1}^{\infty} \ \frac{1}{(1 - \frac{z^2}{n^2})}$$

$$z = \frac{mc\theta}{\pi} = \frac{mc}{\pi\rho}.$$

$$P_0 = mc \ \frac{1}{\sin(\frac{mc}{\rho})} = \rho\prod_{n=1}^{\infty} \ \frac{1}{(1 - \frac{m^2c^2}{n^2\pi^2\rho^2})}. \quad (4.15)$$

Thus, we got the expression for energy proportional to light-momentum ρ, with explicit form when $m = 0$, $P_0(m = 0) = \rho$.

Formula for the momentum. Because,

$$P = mc\cot(mc\theta),$$

we, firstly, write

$$P = \frac{mc}{\pi}\ \pi\cot(\pi z), \quad z = mc\theta\frac{1}{\pi} = \frac{mc}{\pi\ \rho}.$$

$$P = \frac{mc}{\pi}\ (\ \frac{\pi\rho}{mc} + \ \frac{mc}{\pi}\ \sum_{n=1}^{\infty}(\ \frac{1}{\frac{mc}{\pi\ \rho}+n} + \frac{1}{\frac{mc}{\pi\ \rho}-n}\ ).$$

$$P = \rho\ +\ mc\ \sum_{n=1}^{\infty}(\ \frac{1}{\frac{mc}{\rho}+n\pi} + \frac{1}{\frac{mc}{\rho}-n\pi}\ ). \quad (5.16)$$

$P(m=0) = \rho$.

$$P = \frac{1}{\chi}\ +\ mc\ \sum_{n=1}^{\infty}(\ \frac{1}{mc\chi+n\pi} + \frac{1}{mc\chi-n\pi}\ ).$$

On making use of mass-shell equation (1.1) and by taking N instead of ∞, we come to an algebraic relation between mc and ρ, i.e, between inertial mass and light-momentum. For the rest state we have.

$$P = 0, \quad \cot(mc\theta) = 0, \quad mc\theta = \frac{\pi}{2} \pm \pi n = \frac{mc}{\rho}. \quad (4.17)$$

Thus, at the rest ρ is quantized.

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