Computing Entangled Random States Probabilities for Composite $2 \otimes 2$ and $2 \otimes 3$ Systems

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Motivation

The present note is devoted to the measurement theoretical aspects of the *entangled states* [1] characterization. [1] Formulation of an effective computational algorithms for description of quantum entangled states properties is highly important from theoretical positions as well from the computing and communications issues. In order to estimate quantum computational resources of a given system it is reasonable to "count" the entangled states. From the measurement theoretical point of view, (cf. [2],[3]), the "counting" of entangled states corresponds to the determination of a relative volume of entangled states with respect to all possible states. This number gives the geometric probability of entangled states [4]. It turned out, that apart from this theoretical measurement issue, the quantum separability problem has many interesting and deep mathematical aspects, including the computational ones. In particular, it was proved that the problem is computationally NP-hard (cf. [5], [6]) and even for low dimensional quantum systems the calculations of entanglement characteristics are cumbersome. Below we report on our computations of the measurement theoretical characteristics of 2-qubits and qubit-qutrit pairs, particularly present the results of calculations of the geometric probability of the mixed separable/entangled states.

Selecting the separable matrices

The first issue to be addressed is the efficient schemes formulation for selection of the separable density matrices among all possible states. The complete answer to the question, how to find the separable density matrices among a random ensemble of matrices, is unknown for a generic case. However, for two qubits ($2 \otimes 2$) and qubit-qutrit pairs ($2 \otimes 3$), the well-established criterion exists.

• **The Peres-Horodecki criterion** • The famous Peres-Horodecki separability criterion [7],[8] is based on the partial transposition operation. The partial transpose ρ^{T_B} of a density matrix ρ of binary $A \otimes B$ system, with respect to the subsystem B is defined as $\rho^{T_B} = I \otimes T\varrho$, where T stands for the standard transposition operation in subsystem B. According to the Peres-Horodecki a given state ρ in dimensions $2 \otimes 2$ and $2 \otimes 3$ is separable if its *partial transpose* is positive and only then. Unfortunately, this criterion is not a universal. For higher dimensions, there are entangled states with a positive partial transpose (PPT). Even for binary, $3 \otimes 3$ system, one can find the counterexample for the Peres-Horodecki criterion. However, if consideration is restricted by $2 \otimes 2$ and $2 \otimes 3$ systems, the checking of partially transposed matrices on the semi-positivity reduces the selection issue to the correct algebraic problem. Furthermore, it turns that the latter admits the efficient computational formulation.

• **Positivity of the density matrices** • The positive semi-definiteness of an arbitrary Hermitian $n \times n$ matrix means non-negativity of its eigenvalues, $x_k \geq 0\,,\ k = 1, 2, \dots, n$. Since eigenvalues $\{x\}$ are non-polynomial functions of matrix the usage of these inequalities is not computationally effective. Fortunately, for the Hermitian matrix the semi-positivity can be stated as non-negativity of first n-symmetric polynomials in its eigenvalues

$$S_k \geq 0\,, \quad k = 1, 2, \dots, n\,. \tag{1}$$

Symmetric polynomials S_k are expressible as polynomial functions of traces of powers of the density matrix $t_k = \mathrm{Tr}(\rho^k)$, and therefore are very attractive from the computational point of view (cf. for details [9], [10] and references therein).

Geometric probability

Based on the Peres-Horodecki criterion the separability probability for bipartite systems of 2-qubits or qubit-qutrit can be represented as:

$$\mathcal{P}_{\mathrm{sep}}(\mathfrak{P}_+ \cap \widetilde{\mathfrak{P}}_+) = \frac{\int_{\mathfrak{P}_+ \cap \widetilde{\mathfrak{P}}_+} d\mu}{\int_{\mathfrak{P}_+} \mathrm{d}\mu}\,, \tag{2}$$

[1] According to the definition the entangled states of a composite finite dimensional quantum system form a subset of all states complement to the states that are representable by a convex combination of product states, with factors corresponding to each subsystem.

where the integrals in (2) are over the following spaces: $\mathfrak{P}_+$ - is the total space of states, $\widetilde{\mathfrak{P}}_+$ - the image of $\mathfrak{P}_+$ under the partial transposition map $\mathrm{I}\otimes\mathrm{T}$. The intersection $\mathfrak{P}_+\cap\widetilde{\mathfrak{P}}_+$ represents the subset of $\mathfrak{P}_+$ that is invariant under the partial transposition map $\mathrm{I}\otimes\mathrm{T}$:

$$\mathfrak{P}_+\cap\widetilde{\mathfrak{P}}_+ = \{\rho\in\mathfrak{P}_+ \,|\, \mathrm{I}\otimes\mathrm{T}\rho\in\ \mathfrak{P}_+\}\ ,$$

The measure $\mathrm{d}\mu$ in integrals (2) is determined by the Riemannian metrics on the space of density matrices. Below we state results of our evaluation of integrals in (2) for two measures, the Hilbert-Schmidt and Bures.[2] Because straightforward calculations of the multidimensional integral over the semi-algebraic set $\mathfrak{P}_+\cap\widetilde{\mathfrak{P}}_+$ is not a simple task, it is instructive to proceed with the numerical Monte-Carlo methods. Generating the random density matrices and then testing the symmetric polynomials, constructed from the partially transposed matrices ρ^{T_B} , on the Peres-Horodecki conditions

$$S_k^{T_B}\geq 0\,,\qquad k=1,2,\dots,n\,, \tag{3}$$

we will find the relative number of separable states, i.e. determine the probability (2).

The induced measure for mixed states

In our computations we adopt the method of the induced measures (cf. [13, 14]) and consider random density matrices distributed according to the Hilbert-Schmidt and Bures probability measure. Random matrices preparation procedure starts with the *Ginibre ensemble* [15] generation. Let $\mathrm{M}(\mathbb{C},n)$ is the space of $n\times n$ matrices whose entries are complex numbers, distributed as independent standard normal complex random variables

$$p(z_{ij})=\frac{1}{\pi}\exp(-|z_{ij}|)\,,\qquad i,j=1,2,\dots,n\,.$$

Ginibre's measure of this probability distribution for matrix $Z\in\mathrm{M}(\mathbb{C},n)$ is defined as

$$\mathrm{d}\mu_{\mathrm{G}}(Z)=\frac{1}{\pi^{n^2}}\exp\left(-\mathrm{Tr}\left(Z^\dagger Z\right)\right)\mathrm{Tr}\left(\mathrm{d}Z^\dagger\mathrm{d}Z\right)\ . \tag{4}$$

Having the random Ginibre matrices the simple algorithms for generation the Hilbert-Schmidt and the Bures ensembles exists [14].

• **The Hilbert-Schmidt ensemble** • In order to generate the Hilbert-Schmidt states consider a square $n\times n$ complex random matrix Z from the Ginibre ensemble. Introducing, for given Z, the matrix

$$\rho_{\mathrm{HS}}=\frac{Z^\dagger Z}{\mathrm{Tr}\,(Z^\dagger Z)}\,, \tag{5}$$

one can convinced that by construction ϱ is Hermitian, positive definite with unit norm matrix and furthermore represents an element from the Hilbert-Schmidt ensemble.

• **The Bures ensemble** • The $n\times n$ density matrix distributed in accordance with the Bures measure can be generated considering the random matrix of the following form

$$\rho_{\mathrm{B}}=\frac{(\mathbb{I}+U)ZZ^+(\mathbb{I}+U^+)}{\mathrm{Tr}\left[(\mathbb{I}+U)ZZ^+(\mathbb{I}+U^+)\right]}\,. \tag{6}$$

In (6) the complex matrix Z is an element of the Ginibre ensemble, while U is a unitary matrix distributed according to the Haar measure on the unitary group $U(n)$. As it was shown in [14], the probability distribution for matrices ϱ_{B} coincides with the Bures one.

Sufficient conditions for 2-qubits entanglement

Consider the system composed from pairs of qubits $A\otimes B$ in generic 15-parameter mixed state with density matrix written in the so-called Fano form [16]:

$$\rho=\frac{1}{4}\left[\mathbb{I}_4+\boldsymbol{a}\cdot\boldsymbol{\sigma}\otimes\mathbb{I}_2+\mathbb{I}_2\otimes\boldsymbol{\sigma}\cdot\boldsymbol{b}+c_{ij}\,\boldsymbol{\sigma}_i\otimes\boldsymbol{\sigma}_j\,\right]\,.$$

The parameters $\boldsymbol{a}$, and $\boldsymbol{b}$, are Bloch vectors of individual qubits in states ρ_A and ρ_B, determined from ρ by taking the partial traces:

$$\rho_A=\mathrm{Tr}_B\,\rho\,,\qquad \rho_B=\mathrm{Tr}_A\,\rho\,. \tag{7}$$

Nine coefficients c_{ij} are entries of the *"correlation matrix"* $||\mathrm{C}||_{ij}:=c_{ij}$, which comprise an information on correlations occurring between A and B parts. Based on the equations (3) we prove the following statement: *Any density matrix ρ, obeying the inequalities*

$$\det{}^2||\mathrm{M}||>1\,,\qquad \det{}^2||\mathrm{C}||>1\,, \tag{8}$$

with necessity is the entangled matrix. In (8) M stands for the so-called *Schllenz-Mahler matrix*:

$$\mathrm{M}_{ij}:=c_{ij}-a_ib_j\,. \tag{9}$$

The density matrices from the complementary domain

$$-1\leq\det||\mathrm{M}||\leq 1\,,\qquad -1\leq\det||\mathrm{C}||\leq 1\,, \tag{10}$$

[2]Since the volume of space of states in terms of both Hilbert-Schmidt metric [11] and Bures metric [12] is known, the problem of determination of separability probability reduces to the calculation of the integral over $\mathfrak{P}_+\cap\widetilde{\mathfrak{P}}_+$.

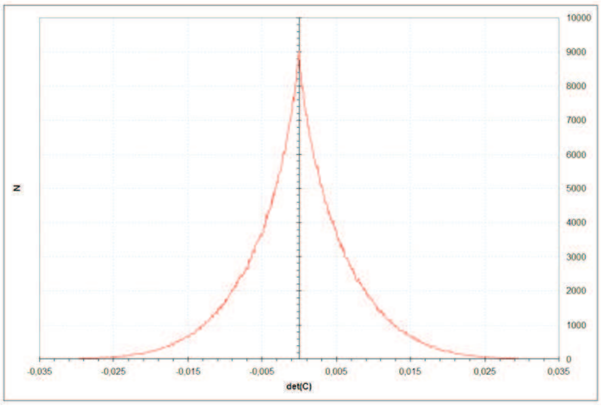

Figure 1: Distribution of separable states with respect to the correlation measure det ||C|| for 10^6 matrices from the Hilbert-Schmidt ensemble.

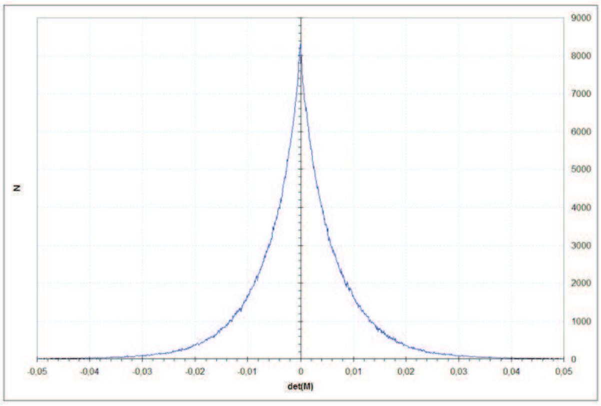

Figure 2: Distribution of separable states with respect to the Schllenz-Mahler entanglement measure det ||M|| for 10^6 random Hilbert-Schmidt matrices.

are separable as well as entangled ones. Using the above described algorithms for the generation of random density matrices we analyze the distribution of separable matrices. Figure 1 and Figure 2 show the character of separable density matrices distributions for 2 qubits with respect to det ||C|| and det ||M||.

Probabilities and conjectures

Finally we give the values of probabilities for 2-qubits and qubit-qutrit composite systems, whose density matrices are distributed according to the the Hilbert-Schmidt and Bures probability measure. The results of our numeric experiments are listed in the Table 1, where the fractional approximations for probabilities are given in the last column. Note that our numerical computations strongly support the fractional value 8/33 for qubits pairs separability probability, that had been conjectured by P.B.

System	Separable	Rational	Primes
H-S metric			
$2 \otimes 2$	24.24 %	$\frac{8}{33}$	$\frac{2^3}{3*11}$
$2 \otimes 3$	3.73 %	$\frac{16}{429}$	$\frac{2^4}{3*11*13}$
Bures metric			
$2 \otimes 2$	7.3 %	$\frac{799}{10843}$	$\frac{799}{7*1549}$
$2 \otimes 3$	0.1 %	$\frac{79}{63499}$	$\frac{79}{11*13*443}$

Table 1: Probabilities for 2-qubits and qubit-qutrit.

Slater few years ago [17]. At the same time, an analytical derivation of this result as well as other simple rational values given in the Table 1 remains yet an interesting unsolved problem.

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