On the Algebraic Structure of Global Orbit Space of Mixed Qudit States

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Introduction and brief statement

The entire information on d-dimensional quantum system is encoded in the space of states $\mathfrak{P}_+$, the space of Hermitian semi-positive $d \times d$ matrices with a unit trace. However, the physically relevant space corresponding to the isolated system is given by the factor space, $\mathfrak{P}_+/\mathrm{U}(d)$, where $\mathrm{U}(d)$ is the unitary group acting in adjoint way on $\mathfrak{P}_+$,

$$(\mathrm{Ad}\, g\,)\varrho = g\,\varrho\, g^{-1}\,. \tag{1}$$

The collection of all $\mathrm{U}(d)$-orbits, together with the quotient topology and differentiable structure defines the "global orbit space", $\mathfrak{P}_+/\mathrm{U}(d)$. The orbit space $\mathfrak{P}_+/\mathrm{U}(d)$ admit description in terms of the elements of integrity basis for the corresponding ring of $\mathrm{U}(d)$-invariant polynomials. Arguably the most effective to achieve this is to use the Processi and Schwarz method, introduced in 80th of last century [1, 2]. According to the Processi and Schwarz the orbit space is identified with the semi-algebraic variety, defined by the syzygy ideal for the integrity basis and the semi-positivity condition of a special, so-called "gradient matrix", $\mathrm{Grad}(z) \geqslant 0$, constructing from the integrity basis elements. In the present report results of application of this generic approach to the construction of $\mathfrak{P}_+/\mathrm{U}(d)$ are stated. In particular, we analyze the algebraic conditions on the elements of the integrity basis of the polynomial ring $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{U}(d)}$ arising from the semi-positivity of Grad$-$ matrix. It turns that these conditions are equivalent to the requirement of Hermicity of the density matrices $\varrho \in \mathfrak{P}_+$ and thus are satisfied identically for any physical states. The conditions $\mathrm{Grad}(z) \geqslant 0$ do not bring new restrictions on elements of the integrity basis for $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{U}(d)}$.

Orbit space $\mathfrak{P}_+/\mathrm{U}(d)$ via the Processi-Schwarz method

Consider a compact Lie group G acting linearly on the real d-dimensional vector space V. Let $\mathbb{R}[V]^{\mathrm{G}}$ is the corresponding ring of the G-invariant polynomials on V. Assume $\mathcal{P} = (p_1, p_2, \dots, p_q)$ is a set of homogeneous polynomials that form the integrity basis,

$$\mathbb{R}[x_1, x_2, \dots, x_d]^{\mathrm{G}} = \mathbb{R}[p_1, p_2, \dots, p_q]\,.$$

Define the polynomial mapping:

$$p : V \to \mathbb{R}^q;\ (x_1, \dots, x_d) \to (p_1, \dots, p_q). \tag{2}$$

Since p is constant on the orbits of G it induces a homeomorphism of the orbit space V/G and the image X of p-mapping; $V/G \simeq X$ [3]. Let $Z \subseteq \mathbb{R}^q$ denote the locus of common zeros of all elements of the *syzygy ideal* $I_{\mathcal{P}}$ of $\mathcal{P}$, then Z is algebraic subset of $\mathbb{R}^q$ such that $X \subseteq Z$. The set X can be described as a semi-algebraic variety of Z defined the following conditions:

a) $z \in Z$, where Z is the surface defined by the syzygy ideal $I_{\mathcal{P}}$ for the integrity basis $\mathcal{P}$;

b) $\mathrm{Grad}(z) \geqslant 0$,

where the $\mathrm{Grad}(z)$ denotes $q \times q$ matrix whose entries are given by the inner products of gradients

$$||\mathrm{Grad}||_{ij} = \left(\mathrm{grad}\,(p_i)\,, \mathrm{grad}\,(p_j)\right)\,.$$

Using these basic ingredients we describe the semi-algebraic structure of the orbit space $\mathfrak{P}_+/\mathrm{U}(d)$ and exemplify the procedure briefly considering three level quantum system. Because the adjoint action (1) is equivalent to the linear action on $\mathbb{R}^{d^2}$

$$\boldsymbol{v}' = L\boldsymbol{v}\,, \qquad L \in \mathrm{U}(d) \otimes \overline{\mathrm{U}(d)}\,,$$

where a line over expression means the complex conjugation one can construct the integrity basis $\mathcal{P}$ in terms of the so-called trace invariants; $\mathbb{R}[v_1, v_2, \dots, v_{d^2}]^{\mathrm{U}(d)} = \mathbb{R}[t_1, t_2, \dots, t_d]$,

$$t_k = \mathrm{tr}\left(\varrho^k\right)\,, \qquad k = 1, 2 \dots, d\,. \tag{3}$$

In terms of (3) the Grad$-$matrix can be easily evaluated

$$\mathrm{Grad} = \begin{pmatrix} d & 2t_1 & 3t_2 & \cdots & dt_{d-1} \\ 2t_1 & 2^2 t_2 & 2\cdot 3t_3 & \cdots & 2\cdot dt_d \\ 3t_2 & 2\cdot 3t_3 & 3^2 t_4 & \cdots & 3\cdot dt_{d+1} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \\ dt_{d-1} & 2\cdot dt_d & 3\cdot dt_{d+1} & \cdots & d^2 t_{2d-2} \end{pmatrix}.$$

Note that all polynomials t_k in Grad−matrix with $k > d$ are expressible as polynomials in $(t_1, t_2, \dots, t_d)$. It turns that it is similar to the "square" of the Vandermonde matrix, Disc $= \Delta\Delta^T$, whose entries are determined by powers of roots $(x_1, x_2, \dots x_d)$ of the characteristic equation:

$$\begin{aligned} &\det ||x - \varrho|| \\ &= x^d - S_1 x^{d-1} + S_2 x^{d-2} - \dots (-1)^d S_d = 0 \,. \end{aligned} \tag{4}$$

Thus the semi-positivity of the Grad−matrix is equivalent to the reality condition of eigenvalues of the density matrix ϱ. Since the density matrices by construction are Hermitian, we conclude that the Procesi-Schwarz inequalities are satisfied identically on $\mathfrak{P}_+$. Summarizing, the orbit space $\mathfrak{P}_+/\mathrm{U}(d)$ is given by the inequalities:

$$S_k \geq 0\,, \qquad k = 1, 2, \dots, d\,. \tag{5}$$

$$\text{Disc } \geq 0\,. \tag{6}$$

Example: the orbit space of qutrit

To exemplify the Procesi-Schwarz method consider the orbit space construction for 3-dimensional quantum system, the qutrit. The integrity basis for U(3)-invariant polynomial consist from t_1, t_2, t_3. For a visibility we consider the case of normalized density matrices, $t_1 = 1$.

It is interesting to note that description of the qutrit orbits is similar to the flavor symmetries analysis of hadron classification within the famous "Eightfold way". The corresponding mathematical issues were elaborated in very elegant way more than forty ears ago by by Michel and Radicati [4] and were adapted to the treatment of quantum states [5], [6], [7].

According to (5)-(6) the U(3)-orbit space is given by the triangle domain A-B-C on Figure 1, bounded by the lines

A-B $\quad t_3 = \frac{1}{18}(-4 + 18t_2 + \sqrt{2}(3t_2 - 1)^{3/2})\,,$

A-C $\quad t_3 = \frac{1}{18}(-4 + 18t_2 - \sqrt{2}(3t_2 - 1)^{3/2})\,,$

B-C $\quad t_3 = \frac{3}{2}t_2 - \frac{1}{2}\,,$

with vertexes $A(\frac{1}{3}, \frac{1}{9})$, $B(1,1)$ and $C(\frac{1}{2}, \frac{1}{4})$.

It is instructive to state the correspondence of these algebraic results with the known classification of orbits with respect to their stability group. Having in mind this issue consider the Bloch parametrization for qutrit

$$\rho = \frac{1}{3}\left(\mathbb{I}_3 + \sqrt{3}\,\boldsymbol{\xi}\cdot\boldsymbol{\lambda}\right), \tag{7}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_8) \in \mathbb{R}^8$ denote the Bloch vector and $\boldsymbol{\lambda}$ is the vector, whose components are

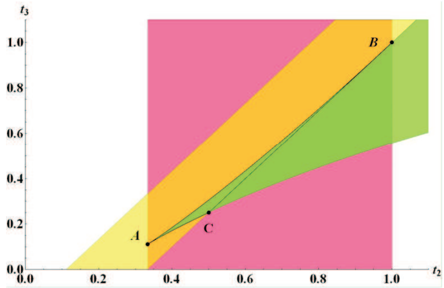

Figure 1: Triangle domain A-B-C as the orbit space of qutrit.

elements $(\lambda_1, \lambda_2, \cdots, \lambda_8)$ of $\mathfrak{su}(3)$ algebra basis, say the Gell-Mann matrices, obeying

$$[\lambda_i, \lambda_j] = 2\imath f_{ijk}\lambda_k\,, \qquad \mathrm{tr}\,(\lambda_i\lambda_j) = 2\delta_{ij}\,. \tag{8}$$

Consider the set of vectors tangent to the orbit passing through a fixed point ϱ:

$$l_i = \imath[\lambda_i, \varrho]\,. \tag{9}$$

By definition he dimension of orbit, $\dim(\mathcal{O}_\varrho)$ is given by the dimension of the tangent space to the orbit, $T_{\mathcal{O}_\varrho}$, and therefore it equals to the number of linearly independent vectors among eight tangent vectors $l_1, l_2, \dots, l_8$. This number depends on the point ϱ and according to the well-known theorem from linear algebra is given by the rank of the so-called Gram matrix

$$A_{ij} = \frac{1}{2}\,\|\mathrm{tr}(l_i l_j)\|\,. \tag{10}$$

In the Bloch parameterization (10) we find that

$$A_{ij} = \frac{4}{3}\,f_{ims}f_{jns}\xi_m\xi_n\,. \tag{11}$$

One can convinced that determination of rank$|A|$ reduces to the evaluation of the rank of the diagonal representative of ϱ_{diag}. For diagonal matrices the Bloch vector is $\boldsymbol{\xi}^{\text{diag}} = (0,0,0,\xi_3,0,0,0,\xi_8)$ and the therefore $|A^{\text{diag}}|$ reads

$$\begin{aligned} A^{\text{diag}} = \frac{1}{3}\,\mathrm{diag}\Big(&4\xi_3^2, 4\xi_3^2, 0, (\xi_3 + \sqrt{3}\xi_8)^2, \\ &(\xi_3 + \sqrt{3}\xi_8)^2, (\xi_3 - \sqrt{3}\xi_8)^2, (\xi_3 - \sqrt{3}\xi_8)^2, 0\Big)\,. \end{aligned} \tag{12}$$

From (12) we conclude that there are orbits of three different dimensions:

- the orbits of maximal dimension, $\dim(\mathcal{O}_\varrho) = 6$,
- the orbits of dimension, $\dim(\mathcal{O}_\varrho) = 4$,
- zero dimensional orbit, one point $\boldsymbol{\xi} = 0$.

These orbits are in agreement with the general orbit classification based on the group of transformations G_ϱ– the isotropy group (or stability group), which stabilize point $\varrho \in \mathcal{O}_\varrho$. The orbits of different dimensions have a different stability groups; for the points lying on the orbit of maximal dimension the stability group is the Cartan subgroup $U(1) \otimes U(1) \otimes U(1)$, while the stability group of points with the diagonal representative λ_8 is $U(2) \otimes U(1)$. The dimensions of listed orbits agree with the general formula $\dim \mathcal{O}_\varrho = \mathrm{dim G} - \mathrm{dim G}_\varrho$. Since the isotropy group of any two points on the orbit are the same up to conjugation, the orbits can be partitioned into sets with equivalent isotropy groups.[1] This set is known as *"strata"*.

Concluding we refer to the relations between certain domains and lines of the triangle A-B-C, depicted on the Figure 1, and the corresponding strata. The domain inside the triangle A-B-C corresponds to the principal strata with the stability group $U(1) \times U(1) \times U(1)$. The discriminant is positive $|\mathrm{Disc}| > 0$ and the density matrix has three different real eigenvalues, the representative matrix reads $\frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\,(\xi_3\lambda_3 + \xi_8\lambda_8))$, with ξ_3 and ξ_8 subject to the constraints (5) and (6) The S_3 coefficient vanishes at line B-C. The boundary line B-C, excluding vertices B and C also belongs to the principal stratum, while points B and C belong to the stratum of lower dimension. On the sides A-B and A-C the discriminant is zero $|\mathrm{Disc}| = 0$, hence, the density matrix has three real eigenvalues and two of them are equal. At point B two eigenvalues of ϱ are zero. The lines (A-B)/{A} and (A-C)/{A} represent the degenerate 4-dimensional orbits whose stability group is $U(2) \otimes U(1)$. Finally, the point A is the zero dimensional stratum corresponding to the maximally mixed state $\varrho = \frac{1}{3}\mathbb{I}_3$. The details of the orbit types are collected in the Table below.

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[1]Note that the isotropy group of a point $\varrho \in \mathcal{O}_\varrho$ depends only on the algebraic multiplicity of the eigenvalues of the matrix ϱ.

$\dim \mathcal{O}$	Stability group	Constraints
6	Interior of triangle A-B-C	$\mathrm{Disc} > 0, S_2 > 0, S_3 > 0$
	Boundary: (B-C)/{B,C}	$\mathrm{Disc} > 0, S_2 > 0, S_3 = 0$
4	Boundary: (A-B)/{A} (A-C)/{A}	$\mathrm{Disc} = 0, S_2 > 0, S_3 \geq 0$
0	Point:{A}	$\mathrm{Disc} = S_2 = S_3 = 0$

Table. Decomposition into strata.

References

[1] C. Procesi and G. Schwarz, The geometry of orbit spaces and gauge symmetry breaking in supersymmetric gauge theories, Phys. Lett. **B 161**, 117-121 (1985).

[2] C. Procesi and G. Schwarz, Inequalities defining orbit spaces, Invent.math. **81** 539-554 (1985).

[3] D. Cox, J, Little and D. O'Shea, Ideals, Varieties, and Algorithms, Third Edition, Springer, (2007).

[4] L.Michel and L.A. Radicati, The geometry of the octet, Ann. Inst. Henri Poincare, Section A, **18**, 185-214 (1973).

[5] M.Adelman, J.V.Corbett and C.A.Hurst, The geometry of state space, Foundation of Physics, **23**, 211-223 (1993).

[6] M. Kus and K. Zyczkowski, Geometry of entangled states, Phys. Rev. **A 63**, 032307, (2008).

[7] L.J.Boya and K.Dixit, Geometry of densty matrix states, Phys. Rev. **A 78**, 042108 (2008).

[8] V. Gerdt, A. Khvedelidze and Yu. Palii, On the ring of local polynomial invariants for a pair of entangled qubits, Zapiski POMI **373**, 104-123 (2009).