STUDIES OF THE WIGNER QUASIPROBABILITY DISTRIBUTIONS

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On the negativity of the Wigner functions for N- level quantum systems

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A long-standing issue of "quantum analogues" of the statistical distributions of classical mechanics consists in a correct definition of the mapping between operators on the Hilbert space of a finite-dimensional quantum system and the so-called quasiprobability distributions [1] defined over the symplectic flag manifold [2, 3].
Guiding by the Weyl-Stratonovich correspondence [4, 5], we propose the construction method of the Wigner function (WF) for an N-level quantum system. We derive algebraic equations for eigenfunctions of the Stratonovich-Weyl (SW) kernel of WF and discuss an arbitrariness of its solution. The presentation is exemplified by considering the WFs for 2, 3 and 4-dimensional quantum systems. The results of analytical and numerical computations of the WF characteristics with various SW kernels, including the probability for the WFs from the corresponding Hilbert-Schmidt ensembles to take negative values will be given.

Introduction

The Wigner function $W_\varrho(\Omega_N)$ of an N-dimensional quantum system,

$$W_\varrho(\Omega_N) = \operatorname{tr}\left[\varrho\,\Delta(\Omega_N)\right]\,,$$

is constructed from the density matrix ϱ expanded over the basis of $\mathfrak{su}(N)$ algebra $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_{N^2-1}\}$, with the (N^2-1)-dimensional Bloch vector $\boldsymbol{\xi}$:

$$\varrho = \frac{1}{N}\left[I + \sqrt{\frac{N\,(N-1)}{2}}\,(\boldsymbol{\xi}\,,\boldsymbol{\lambda})\right],$$

and the SW kernel $\Delta(\Omega_N)$ defined over the phase space Ω_N. According to the SW correspondence, $\Delta(\Omega_N)$ should be compatible with:

1. **Reconstruction postulate**: the state ϱ can be reconstructed from the WF (1) as
$$\varrho = \int_{\Omega_N} \mathrm{d}\Omega_N\, \Delta(\Omega_N) W_\varrho(\Omega_N)\,; \qquad (1)$$
2. **Hermicity postulate**:
$$\Delta(\Omega_N) = \Delta(\Omega_N)^\dagger\,; \qquad (2)$$
3. **Finite norm postulate**: the state norm is given by the integral of the Wigner distribution
$$\operatorname{tr}[\varrho] = \int_{\Omega_N} \mathrm{d}\Omega_N W_\varrho(\Omega_N)\,, \qquad \int_{\Omega_N} \mathrm{d}\Omega_N\,\Delta(\Omega_N) = 1\,; \qquad (3)$$
4. **Covariance postulate**: the unitary transform $\varrho' = U(\alpha)\varrho U^\dagger(\alpha)$ induce the kernel change
$$\Delta(\Omega_N') = U(\alpha)^\dagger \Delta(\Omega_N) U(\alpha)\,. \qquad (4)$$

Here Ω_N denotes the flag manifold $\mathbb{F}^N_{d_1,d_2,\dots,d_r}$, where $(d_1, d_2, \dots, d_r, \sum d_i = N)$ is a sequence of positive integers determined from the algebraic multiplicity of SW kernel, and $\mathrm{d}\Omega_N$ agrees with the Haar measure on the $SU(N)$ group up to a constant.

In the article [3], based on the postulates (1)-(4), the following "master equations" have been derived:

$$\operatorname{tr}\left[\Delta(\Omega_N)\right] = 1\,, \qquad \operatorname{tr}[\Delta(\Omega_N)^2] = N\,. \qquad (5)$$

Using solutions to the "master equations" (5), we write down the singular value decomposition (SVD) for $\Delta(\Omega_N)$ with descending order of its eigenvalues

$$\Delta(\Omega|\boldsymbol{\nu}) = U(\Omega)\left[\frac{I}{N} + \sqrt{\frac{(N^2-1)}{2N}}\sum_{\lambda\in H}\mu_s(\boldsymbol{\nu})\lambda_s\right]U(\Omega)^\dagger. \qquad (6)$$

In (6) diagonalizing unitary matrix is $U(\Omega_N)$ and the diagonal matrix is expanded over the maximal Abelian subalgebra $H \subset SU(N)$.

The spectrum of $\Delta(\Omega)$ is constrained: $\mu_3^2(\boldsymbol{\nu}) + \cdots + \mu^2_{N^2-1}(\boldsymbol{\nu}) = 1$, and moduli parameters $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_{N-2})$ distinguish $N-2$ unitary non equivalent Wigner functions

$$W^{(\boldsymbol{\nu})}_{\boldsymbol{\xi}}(\Omega_N) = \frac{1}{N}\left[1 + \frac{N^2-1}{\sqrt{N+1}}(\boldsymbol{n}, \boldsymbol{\xi})\right], \qquad (7)$$

where $\boldsymbol{n} = \sum_3^{N^2-1}\mu_s(\boldsymbol{\nu})\boldsymbol{n}^{(s)}$ and $n_\mu^{(s^2-1)} = \frac{1}{2}\operatorname{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right)$.

Negativity of the Wigner functions

The WF (7) is not a proper distribution; it is certainly non negative for states the Bloch vectors of which lie inside the ball of radius $r_*(N) = \sqrt{N+1}/(N^2-1)$

$$0 \le \boldsymbol{\xi}^2 \le r_*^2(N)\,,$$

while in complementary domain $r_*^2(N) < \boldsymbol{\xi}^2 \le 1$, the WF can take negative values.

In order to quantitatively characterize the deviation of WFs from classical distributions, we generate a random quantum state, build the corresponding WFs and compute the ratio:

$$\mathcal{P}^{(-)}(N) := \frac{\textit{Number of states with negative WF}}{\textit{Total number of generated states}}$$

The quantity $\mathcal{P}^{(-)}(N)$ has a meaning of the "negativity probability", i.e., the probability for WFs to take negative values for a given ensemble of random states of N-level system.

Generating the Hilbert-Schmidt ensemble

The algorithm for the generation of density matrices from the Hilbert-Schmidt ensemble (HSE) of an N-level quantum system includes two steps:

- Generate $N \times N$ matrices Z from the complex Ginibre ensemble, i.e. the family of matrices in which each entry is an independent complex Gaussian random variable of mean zero and variance one;
- Using Z and its complex conjugate $Z^\dagger$, compute the density matrices
$$\varrho_{\mathrm{HS}} = \frac{ZZ^\dagger}{\operatorname{tr}\left(ZZ^\dagger\right)}$$
describing states from the Hilbert-Schmidt ensemble.

Qubit – two level quantum system

The master equations (5) for two level quantum system, the qubit, determine spectrum of SW kernel $\Delta^{(2)}$ uniquely:

$$\operatorname{spec}\left(\Delta^{(2)}\right) = 1/2\left\{(1+\sqrt{3})\,,\ (1-\sqrt{3})\right\}.$$

The states WFs of which are negative, lie outside the Bloch ball of radius $r_*(2) = 1/\sqrt{3}$. The "negativity probability" of qubit from HSE is

$$\mathcal{P}^{(-)}(2) = 0.115\,.$$

Qutrit – three level quantum system

For a 3-level system there is 1-parameter family of SW kernels

$$\operatorname{spec}\left(\Delta^{(3)}(\nu)\right) = 1/2\left\{1-\nu+\delta\,,\ 1-\nu-\delta\,, 2\nu\right\}$$

with $\delta = \sqrt{(1+\nu)(5-3\nu)}$ and $\nu \in (-1, -\frac{1}{3})$.

The edge points $\nu = -1$, $\nu = -\frac{1}{3}$ represent two degenerate kernels

$$\operatorname{spec}\left(\Delta^{(3)}(-1)\right) = \{1, 1, -1\}\,, \quad \operatorname{spec}\left(\Delta^{(3)}(-\frac{1}{3})\right) = \frac{1}{3}\{5, -1, -1\}\,.$$

while point $\nu = (1-\sqrt{5})/2$ corresponds to a singular kernel

$$\operatorname{spec}\left(\Delta^{(3)}_{\text{sing}}\right) = 1/2\left\{1+\sqrt{5}\,, 0\,, 1-\sqrt{5}\right\}.$$

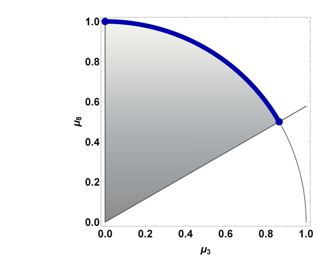

Figure 1: The moduli space of qutrit represented by arc of a unit circle. The angle $\zeta \in [0, \pi/3]$ is related to the parameter ν in (6) labeling unitary nonequivalent WFs, $\nu = \frac{1}{3} - \frac{4}{3}\cos(\zeta)$.

The "negativity probability" of the WF for the qutrit from HSE is

$$\mathcal{P}^{(-)}(3) = 0.137\,.$$

The distribution of WF negativity $\mathcal{P}^{(-)}(\xi)$ with respect to the qutrit Bloch radius ξ for three different SW kernels is depicted in the Fig. 2; two degenerate and one singular ones. The "negativity probability" of qutrit WF as the function of moduli parameter $\zeta = \arccos(\frac{1-3\nu}{4})$ for generic kernel is depicted in the Fig. 3 .

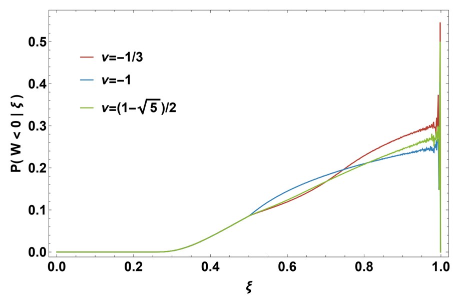

Figure 2: Negativity probability dependence on Bloch radius for degenerate and singular kernels.

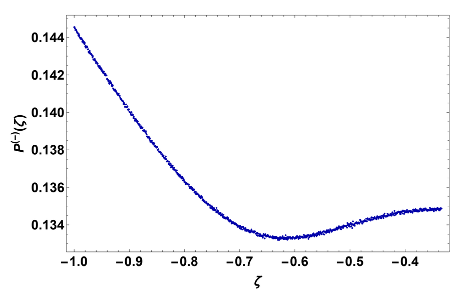

Figure 3: Negativity probability of HSE qutrit WF as function of moduli parameter ν .

Quatrit – four level quantum system

The master equations (5) for a quatrit determine 2-parameter family of SW kernels. The spectrum of $\Delta^{(4)}(\nu_1, \nu_2)$ is determined by points on a unit 2-sphere, $\mu_3^2(\boldsymbol{\nu}) + \mu_8^2(\boldsymbol{\nu}) + \mu_{15}^2(\boldsymbol{\nu}) = 1$ satisfying inequalities:

$$\mu_3 \ge 0\,, \quad \mu_8 \ge \frac{\mu_3}{\sqrt{3}}\,, \quad \mu_{15} \ge \frac{\mu_8}{\sqrt{2}}\,.$$

Constraints (8) define the moduli space of quatrit SW kernel as one out of 24 possible spherical triangles which tessellate a unit sphere and the angles of which are $(\pi/2\,, \pi/3\,, \pi/3)$. Such a triangle is known as the Möbius Triangle with tetrahedral symmetry (Fig. 4) .

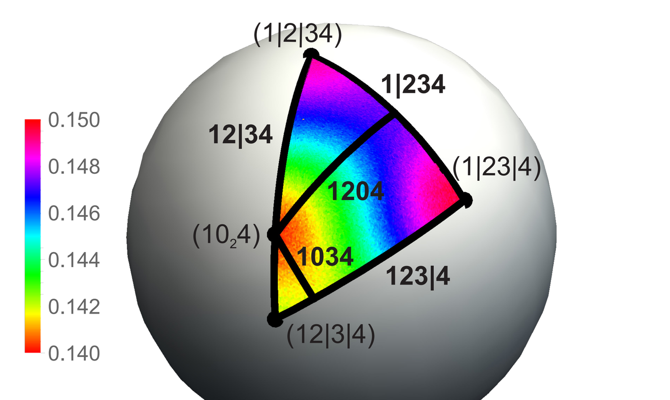

Figure 4: Demonstration of the distribution of WF negatvity probability over the quatrit moduli space represented by the Möbius spherical triangle $(2, 3, 3)$ on a unit sphere. The colorbar identifies colors with the range of WF negatvity probability and the SW kernels with a degenerate spectrum like $\pi_1 > \pi_2 = \pi_3 > \pi_4$ are marked by symbol $(12|34)$ following V.I.Arnold's notation.

Multilevel quantum system

The results of our studies of WF negativity with maximal singularity SW kernels, i.e. kernels spectrum of which have $N-2$ vanishing eigenvalues, are given in the (Fig. 5) .

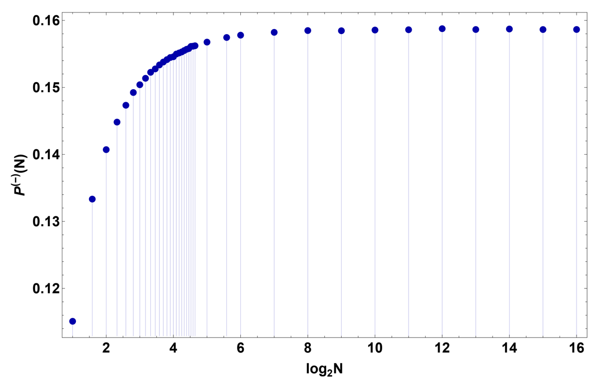

Figure 5: The negativity probability of WF for N-level system with Hilbert-Schmidt states calculated with the maximally singular SW kernel. With growing number of levels $N \to \infty$, the negativity probability tends to the value $\mathcal{P}^{(-)}(\infty) = 0.158655$.

Conclusions

1. The "master equations" for SW kernel $\Delta(\Omega)$ are derived;
2. Ambiguity in the solution to "master equations" is identified and the moduli space of the WF is determined;
3. Probabilistic characteristics of WF negativity are presented.

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