Studies of Wigner Quasi-probability Distribution Functions

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Overview

1. **Formulation of the problems and the general set up**
   - The main objective
   - Standard form of the Wigner function
   - The Stratonovich-Weyl correspondence
2. **Constructing the kernel for Wigner distribution**
   - Master equations
3. Qubit
   - Qubit kernel and the Wigner function
4. Probability of negativity as a measure of non-classicality of space of states

The main objective

We are interested in phase space representations for finite dimensional quantum systems

THE GOAL:

1. Is it possible *to describe the whole family of the WF* $W_\varrho(\Omega|\nu)$ over the classical phase space (Ω) for a generic N-level quantum system with density matrix ϱ? $(+)$
2. What's *the nature of negative values* of quasiprobability distribution functions? (?)
3. Can the non-classicality of a states be described by the negative values of the Wigner function
4. ...

The standard form of the Wigner function

For a given state, describing by the density operator ϱ, the Wigner function $W(\boldsymbol{q}, \boldsymbol{p})$ defined on a classical $2n$-dimensional phase space spanned by the canonical coordinates $\boldsymbol{q}$ and momentum $\boldsymbol{p}$ reads

$$W(\boldsymbol{q}, \boldsymbol{p}) := \int \mathrm{d}^n \boldsymbol{z} e^{\frac{\imath}{\hbar} \boldsymbol{z}\boldsymbol{p}} \langle q + \frac{z}{2} \,|\varrho|\, q - \frac{z}{2} \rangle \,.$$

QPD's

Quasi-Probabilty Distributions – "quantum analogue" of the statistical distribution on the phase space of a classical system

Representation for W via displacement operator D

$$W = \operatorname{Tr} \left[\varrho D \Pi D^{\dagger} \right]$$

where D and Π are the displacement and parity operators respectively.

Symbols and Quasi probability distributions

- For any operator A on the Hilbert space $\mathcal{H}$ of quantum system one can define a family of functions $F_A(\Omega;\nu)$ onto the phase space Ω. Here, ν labels the parameters fixing the function.
- When the operator A represents the density matrix, $A=\varrho$, the corresponding phase-space functions $F_\varrho(\Omega;\nu) := P(\Omega;\nu)$ are named as **Quasiprobability Distributions**.

The Stratonovich-Weyl correspondence

The physically motivated properties of $P(\Omega;\nu)$ were formulated by R.L.Stratonovich more than sixty years ago (1955) and are usually referred to as the Stratonovich-Weyl correspondence

The Stratonovich-Weyl Correspondence

Clauses of SW correspondence:

- **Mapping** • For a density matrix ϱ the Wigner function W_ρ on the classical phase-space (Ω) is given by the map:

$$W_\rho(\Omega) = \mathrm{tr}\left(\rho\Delta(\Omega)\right)$$

defined by the Hermitian kernel $\Delta(\Omega) = \Delta(\Omega)^\dagger$, with a unit norm

$$\int_\Omega \mathrm{d}\Omega\, \Delta(\Omega) = 1$$

- **Reconstruction** • The state ρ can be reconstructed as

$$\rho = \int_\Omega \mathrm{d}\Omega\, \Delta(\Omega) W_\rho(\Omega)\,.$$

- **Covariance** • The unitary symmetry $\rho' = U(\alpha)\rho U^\dagger(\alpha)$ induces the kernel transformation: $\Delta(\Omega') = U(\alpha)^\dagger \Delta(\Omega) U(\alpha)$

The Wigner distribution kernel

The Wigner distribution $W_\varrho(\Omega)$ over a phase space parametrized by the set Ω is determined by the kernel $\Delta(\Omega|\nu)$:

$$W_\varrho^{(\boldsymbol{\nu})}(\vartheta_1, \vartheta_2, \ldots, \vartheta_{d_{\mathbb{F}}}) = \mathrm{tr}\left[\varrho\, \Delta(\Omega|\nu)\right] = \mathrm{tr}\left[\varrho\, X P^{(N)}(\boldsymbol{\nu}) X^\dagger\right] ,$$

Here $P(\nu) = \mathrm{diag}||\pi_1, \pi_2, \ldots, \pi_N||$.

In accordance with the SU(n)-covariance of kernel we identify:

d_F - parameters of unitary matrix $U(\theta) \in SU(N)$ with the coordinates of classical phase-space, $\Omega = (\theta_1, \ldots \theta_{d_F})$.

Deriving the Master equations for $\Delta(\Omega)$

- **Step 1** • The SU(N) symmetry allows to define the "reconstruction" integral for ϱ over the SU(N) group with the Haar measure:

$$\varrho = Z_N^{-1} \int_{SU(N)} \mathrm{d}\mu_{SU(N)}\, \Delta(\Omega_N)\, \mathrm{tr}\left[\varrho \Delta(\Omega_N)\right] .$$

- **Step 2** • Substitute decomposition $\Delta = U(\theta) P U^\dagger(\theta)$ into the identity, after fixing

$$\pi_1 \geq \pi_2 \geq \cdots \geq \pi_N .$$

and evaluate the integral using the Weingarten formula:

$$\int d\mu U_{i_1 j_1} U_{i_2 j_2} \bar{U}_{k_1 l_1} \bar{U}_{k_2 l_2} = \frac{1}{N^2-1} \left(\delta_{i_1 k_1}\delta_{i_2 k_2}\delta_{j_1 l_1}\delta_{j_2 l_2} + \delta_{i_1 k_2}\delta_{i_2 k_1}\delta_{j_1 l_2}\delta_{j_2 l_1}\right)$$

$$-\frac{1}{N(N^2-1)} \left(\delta_{i_1 k_1}\delta_{i_2 k_2}\delta_{j_1 l_2}\delta_{j_2 l_1} + \delta_{i_1 k_2}\delta_{i_2 k_1}\delta_{j_1 l_1}\delta_{j_2 l_2}\right) .$$

Deriving the Master equations for $\Delta(\Omega)$

$$(\mathrm{tr}[P])^2 = Z_N N\,, \quad \mathrm{tr}[P^2] = Z_N N^2\,.$$

- **Step 3** • Fixing Z_N : Standardization

$$Z_N^{-1} \int \mathrm{d}\mu_{SU(N)} W_A^{(\nu)}(\Omega_N) = \mathrm{tr}[A]\,,$$

is satisfied iff $tr\,[P] = Z_N\,N$, resulting in $Z_N = \frac{1}{N}$ and

master equations

$$\mathrm{tr}\,[\Delta(\Omega_N)] = 1\,, \qquad \mathrm{tr}[\Delta(\Omega_N)^2] = N\,.$$

In $\mu_1, \mu_2, \dots \mu_{N^2-1}$ orthonormal basis of $\mathfrak{su}(N)$

$$\Delta(\Omega_N \,|\, \boldsymbol{\nu}) = \frac{1}{N} U(\Omega_N) \left[I + \sqrt{\frac{N(N^2-1)}{2}} \sum_{\lambda \in H} \mu_s(\boldsymbol{\nu}) \lambda_s \right] U^\dagger(\Omega_N)\,,$$

with coefficients $\mu_s(\nu)$ defined on an unit sphere $\mathbb{S}_{N-2}(1)$.

The Wigner function

for N dimensional quantum system with $N-1$ dimensional Bloch vector $\boldsymbol{\xi}$

$$W^{(\boldsymbol{\nu})}_{\boldsymbol{\xi}}(\theta_1, \theta_2, \dots, \theta_d) = \frac{1}{N} \left[1 + \frac{N^2-1}{\sqrt{N+1}} (\boldsymbol{n}, \boldsymbol{\xi}) \right]\,,$$

$$\boldsymbol{n} = \mu_1 \boldsymbol{n}^{(1)} + \mu_2 \boldsymbol{n}^{(2)} + \cdots + \mu_{N-1} \boldsymbol{n}^{(N-1)}\,,$$

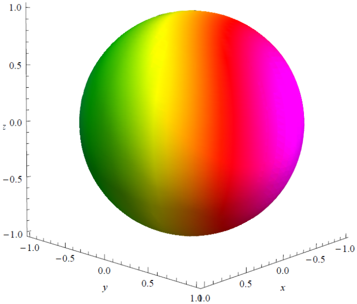
$$\boldsymbol{n}^{(s)}_\mu = \frac{1}{2} \operatorname{tr} \left(U \lambda_s U^\dagger \lambda_\mu \right)\,, \qquad \mu = 1, 2, \dots, N^2-1\,.$$

Qubit kernel and the Wigner function

For a 2-level system the uniquely defined kernel is $P^{(2)} = \frac{1}{2} diag||1+\sqrt{3}, 1-\sqrt{3}||$.

Taking into account that $X = \exp\left(i\,\frac{\alpha}{2}\,\sigma_3\right)\,\exp\left(i\,\frac{\beta}{2}\,\sigma_2\right)\,\exp\left(-i\,\frac{\alpha}{2}\,\sigma_3\right)$, for a qubit parametrized in a standard way by a Bloch vector $\boldsymbol{r}(\psi,\phi) = (r\sin\psi\cos\phi, r\sin\phi\sin\phi, r\cos\psi)$ as $\varrho = \frac{1}{2}(I + \boldsymbol{r}\cdot\boldsymbol{\sigma})$,

$$W_{\boldsymbol{r}}(\alpha,\beta) = tr\left[\varrho\, X\, P^{(2)}\, X^{\dagger}\right] = \frac{1}{2} - \frac{\sqrt{3}}{2}\,(\boldsymbol{r}(-\psi,-\phi), \boldsymbol{n})\ .$$



Probabolity of negativity

The normalized measure $\mathrm{M}_\rho = \frac{1}{Vol_{\Omega_{d_F}}}\mathrm{M}\left\{\Omega \,|\, W_\rho(\Omega) < 0\right\}$ of the unified domain where the Wigner function acquires negative values is a measure of non-classicality (e.g. exhibition of pure quantum correlations).

"How much" non-classicality may be found in uniformly covered set of the states of an N-level system?

$$\mathcal{P} = \frac{1}{Vol(\text{Space of states})} \int \mathrm{M}_\rho \, dV_{H-S}(\rho).$$

Claim

The limit of total non-classicality $\lim_{N\to\infty} \mathcal{P} = \frac{1}{2} erfc\left(\frac{1}{\sqrt{2}}\right) = 0.15866\cdot$, and doesn't depend on the choice of Stratonovich-Weeyl kernel.

Conclusions

1. It is shown that the **kernel** $\Delta(\Omega)$ **satisfies** two algebraic "**master equations**";

2. An ambiguity in the solution to those "master equations" has been analyzed and the **moduli space** of the Wigner quasiprobability distribution **was determined**;

3. The positivity of the WF has been studied and the **probabilistic characteristics of negativity** of the WF **were found**.

4. The **total non-classicality for infinite level system has been found**.

Thank you!