STUDIES OF THE WIGNER QUASIPROBABILITY DISTRIBUTIONS

Describing quantumness of qubits and qutrits by Wigner function's negativity

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A basic idea:

The Wigner function takes negative values for some quantum states. Interpreting this as an evidence of the deviation from classicality, one can define several measures of “quantumness of states” within the phase-space formulation of quantum mechanics of finite-dimensional systems.

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Objective

Wigner quasiprobability distribution (WF) $W(\Omega_N) = \mathrm{tr}\,[\varrho\,\Delta(\Omega_N)]$:

- density matrix $\varrho \in \mathfrak{P}_N$: $\varrho = \varrho^\dagger\,,\ \varrho \geq 0\,,\ \mathrm{tr}\,(\varrho) = 1\,;$
- Stratonovich-Weyl kernel

$\Delta(\Omega_N) \in \mathfrak{P}^*_N$: $\Delta = \Delta^\dagger\,,\ \mathrm{tr}\,(\Delta) = 1\,,\ \mathrm{tr}\,\left(\Delta^2\right) = N\,.$ [1]

Global indicator of classicality:

$$\mathcal{Q}_N = \frac{\text{Volume of orbit subspace } \mathcal{O}[\mathfrak{P}^{(+)}_N]}{\text{Volume of orbit space } \mathcal{O}[\mathfrak{P}_N]}\,,$$

where $\mathcal{O}[\mathfrak{P}^{(+)}_N]$ is the unitary orbit space of states with non-negative WF.

Kenfack-Życzkowski (KZ)-indicator of non-classicality:

$$\delta_N = \int_{\Omega_N} \mathrm{d}\Omega_N \big|W(\Omega_N)\big| - 1\,.$$

[1] *On families of Wigner functions for N-level quantum systems*, V. Abgaryan, A. Khvedelidze, https://arxiv.org/pdf/1708.05981.pdf (2018).

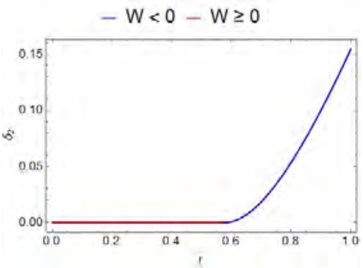
Quantumness of a single qubit

KZ-indicator for a single qubit is zero for the Bloch radius $r \in [0, \frac{1}{\sqrt{3}}]$:

$$\delta_2 = \theta[r - \frac{1}{\sqrt{3}}] \left(\frac{3r^2 + 1}{2\sqrt{3}r} - 1 \right) .$$

Global $\mathcal{Q}$-indicator for a qubit:

$$\mathcal{Q}_2[\mathrm{g_{HS}}] = \frac{1}{3\sqrt{3}} .$$

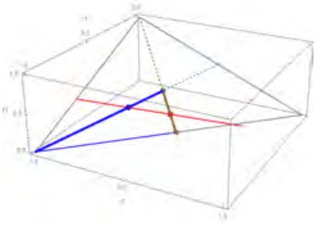


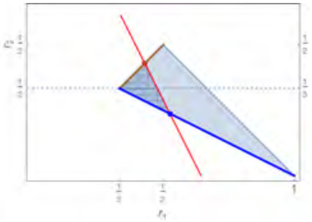
The Wigner function of a qubit is positive definite inside the Bloch ball of radius $r_*(2) = 1/\sqrt{3}$.

Quantumness of a single qutrit

The lower bound of WF determines the positivity region

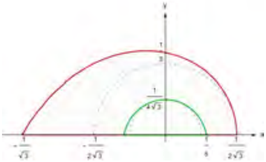
$\sum_{i=1}^{N} \pi_i r_{N-i+1} \leq W(\Omega_N) \leq \sum_{i=1}^{N} \pi_i r_i$





The state space of a qutrit is divided into bands:

$\zeta = 0$



$\zeta = \pi/6$

$\zeta = \pi/3$

Qutrit KZ-indicator for $\zeta = 0$:

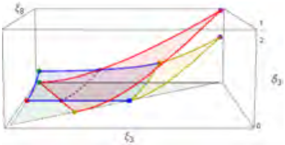
$$\delta_3^{(0)} = \theta[2\sqrt{3}\xi_3 + 2\xi_8 - 1]\,\theta[\sqrt{3}\xi_8 - \xi_3] \times$$
$$\theta[1 - 2\xi_8]\,\theta[1 - 8\xi_8]\,\frac{2\left(\left(\sqrt{3}\xi_3 + \xi_8\right) - \frac{1}{2}\right)^3}{9\xi_3\left(\xi_3 + \sqrt{3}\xi_8\right)}\,.$$

Qutrit KZ-indicator for $\zeta = \pi/3$:

$$\delta_3^{(\pi/3)} = \theta[4\xi_8 - 1]\,\theta[1 - 2\xi_8]\,\big[\,\theta[2\sqrt{3}\xi_3 - 2\xi_8 - 1] \times$$
$$\theta[\sqrt{3}\xi_8 - \xi_3]\left(\frac{2\left((\sqrt{3}\xi_3 + \xi_8) + \frac{1}{2}\right)^3}{9\xi_3(\xi_3 + \sqrt{3}\xi_8)} - 2\right) +$$
$$\theta[\xi_3]\,\theta[2\xi_8 - 2\sqrt{3}\xi_3 + 1]\left(\frac{-(4\xi_8 - 1)^3}{18\left(\xi_3^2 - 3\xi_8^2\right)}\right)\big]\,.$$

Qutrit $\mathcal{Q}$-indicator of classicality:

$$\mathcal{Q}_3[g_{\mathrm{HS}}] = \frac{20\cos^2(\zeta-\pi/6)+1}{128\left(4\cos^2(\zeta-\pi/6)-1\right)^5}\,,$$
$$\min_{\zeta\in[0,\frac{\pi}{3}]}\mathcal{Q}_3(\zeta) = \mathcal{Q}_3\left(\frac{\pi}{6}\right) = \frac{7}{2^7\,3^4}\,.$$



Qutrit KZ-indicators $\delta_3^{(0)}$ (red surface) and $\delta_3^{(\frac{\pi}{3})}$ (blue and yellow surfaces) as functions of two invariants ξ_3 and ξ_8 .

Qutrit global indicator in Hilbert-Schmidt metric.

Some concluding remarks

The global $\mathcal{Q}$-indicator [2] is sensitive to

1. negativity of the Wigner function;
2. local quantum uncertainty in the form of geometric measure on the orbit space.

The KZ-indicator [3] points to the existence of three classes of states:

1. "absolutely classical": $\delta = 0$ for all values of moduli parameters ζ;
2. "absolutely quantum": δ depends on ζ but is never zero;
3. "relatively quantum-classical" states whose classicality is susceptible to a Wigner function representation.

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[2] *The global indicator of classicality of an arbitrary N-level quantum system*, Abgaryan, A. Khvedelidze, A. Torosyan, J Math Sci 251, 301-314 (2020).

[3] *Kenfack-Życzkowski indicator of nonclassicality for two non-equivalent representations of Wigner function of qutrit*, V. Abgaryan, A. Khvedelidze, A. Torosyan, https://arxiv.org/pdf/2009.00375.pdf (2020).