Numerical Elimination Identification of Singularities in Multidimensional Improper Integrals of Second Type

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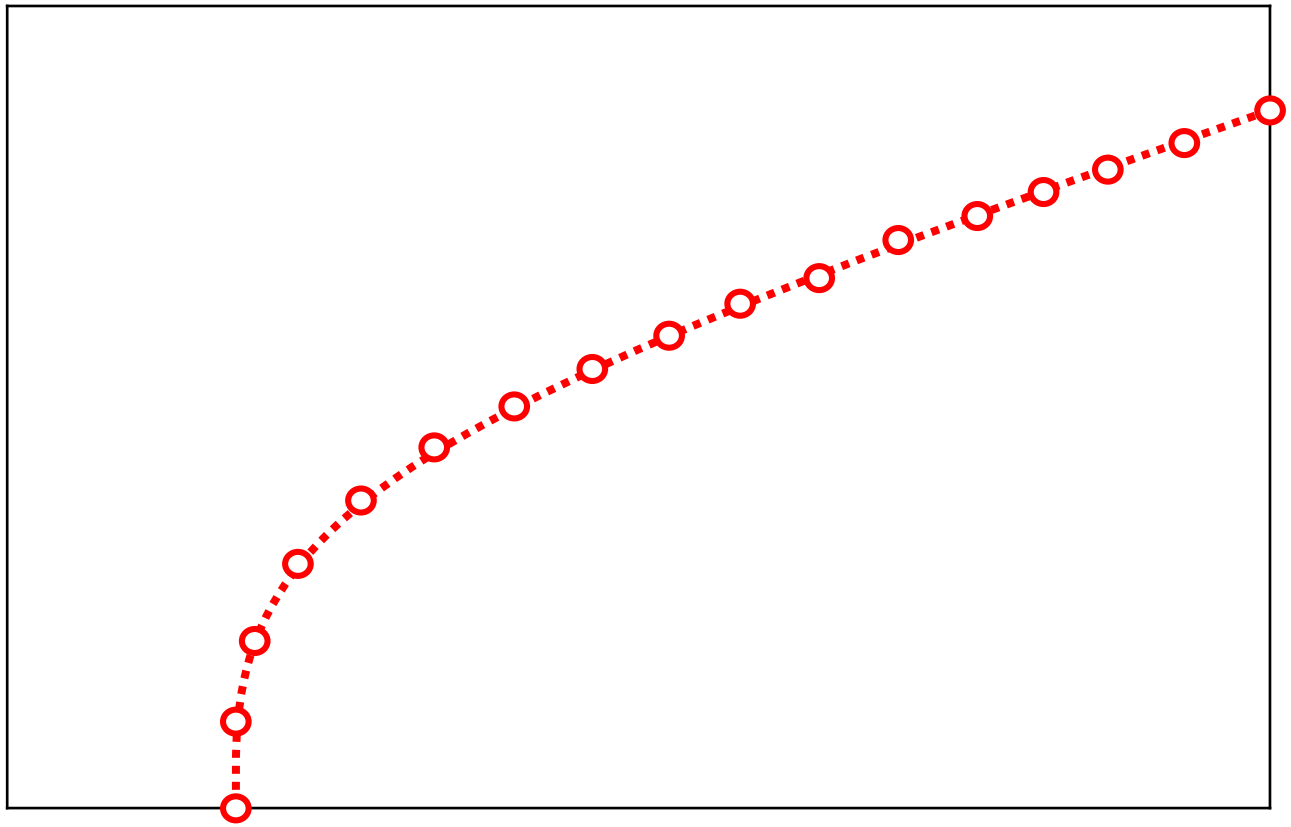
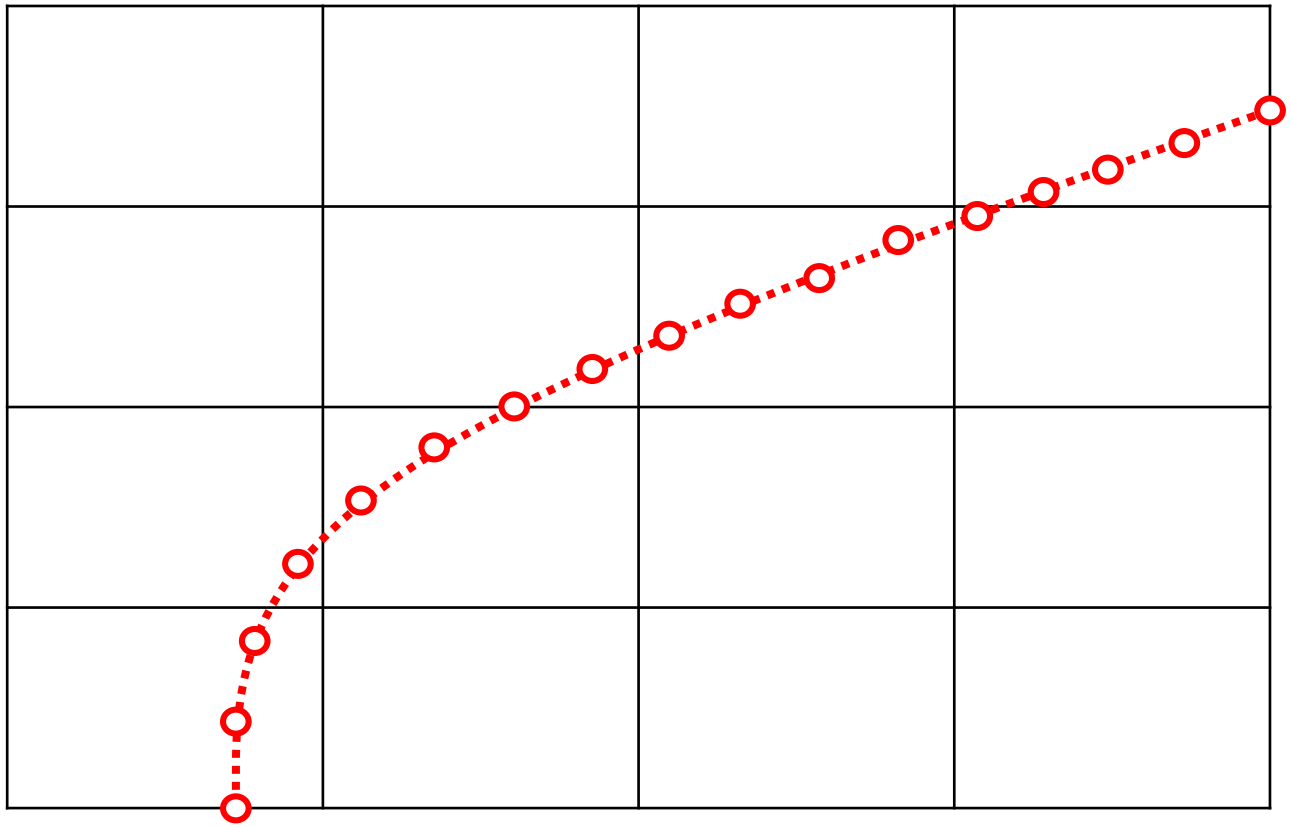
Main idea

$$I = \int_a^b dx \frac{f(x)}{x-c} = \lim_{\varepsilon \to 0} \left\{ \int_a^{c-\varepsilon} dx \frac{f(x)}{x-c} + \int_{c+\varepsilon}^b dx \frac{f(x)}{x-c} \right\}$$

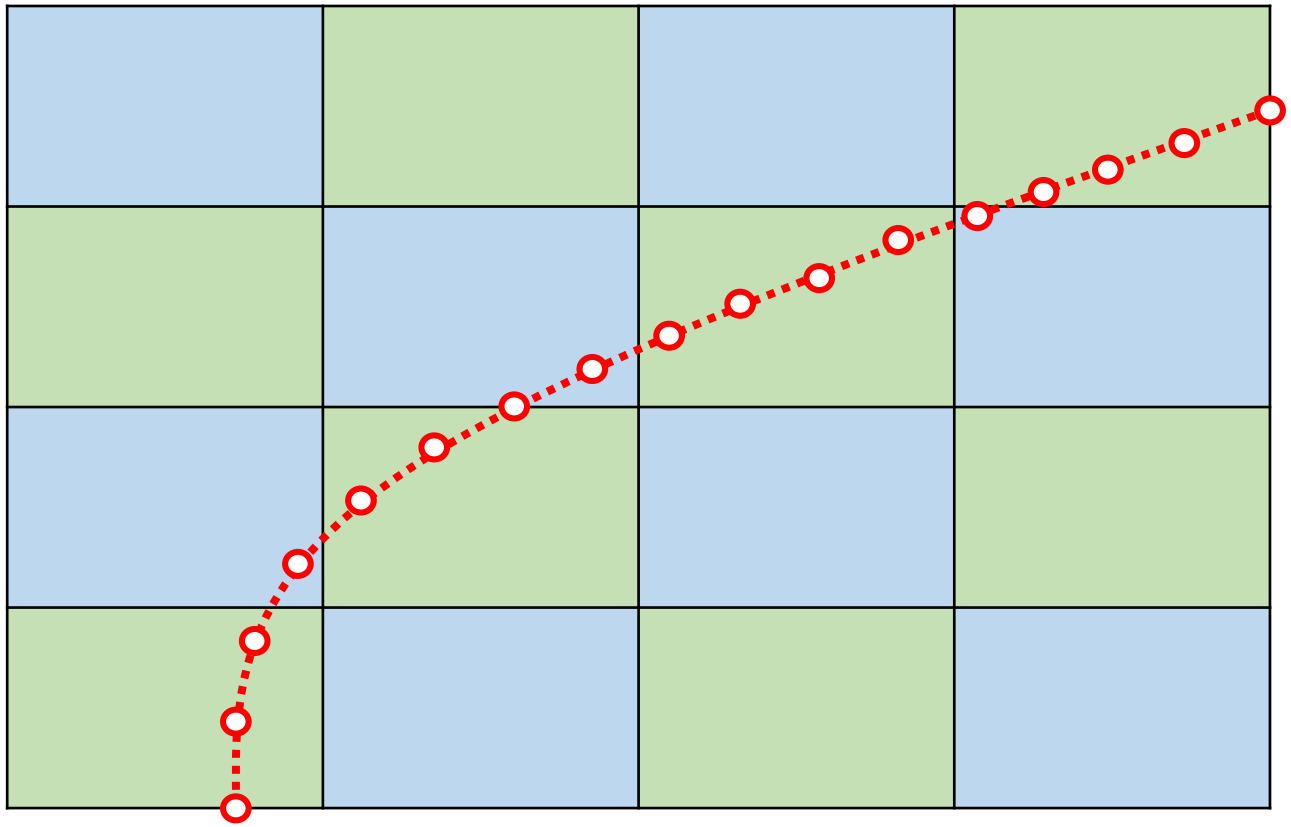
Determining the location of a singularity



Subdivides the integration region into subsegments



Calculate subsegments



Monte-Carlo VEGAS integration

$I = V \frac{\sum f(x)}{N}$	Integration
$D\xi = \frac{\sum f(x)^2}{N} - \left(\frac{\sum f(x)}{N}\right)^2$	variance
$\sigma = V \frac{D\xi}{\sqrt{N}}$	error

I_1 σ_1	I_2 σ_2	I_3 σ_3	I_4 σ_4
I_5 σ_5	I_6 σ_6	I_7 σ_7	I_8 σ_8
I_9 σ_9	I_{10} σ_{10}	I_{11} σ_{11}	I_{12} σ_{12}
I_{13} σ_{13}	I_{14} σ_{14}	I_{15} σ_{15}	I_{16} σ_{16}

I_1 σ_1	I_2 σ_2	I_3 σ_3	I_4 σ_4
I_5 σ_5	I_6 σ_6	I_7 σ_7	I_8 σ_8
I_9 σ_9	I_{10} σ_{10}	I_{11} σ_{11}	I_{12} σ_{12}
I_{13} σ_{13}	I_{14} σ_{14}	I_{15} σ_{15}	I_{16} σ_{16}

$$I = \sum_{i=1}^{n} I_i$$

$$\sigma = \sqrt{\sum_{i=1}^{n} \sigma_i}$$

Test

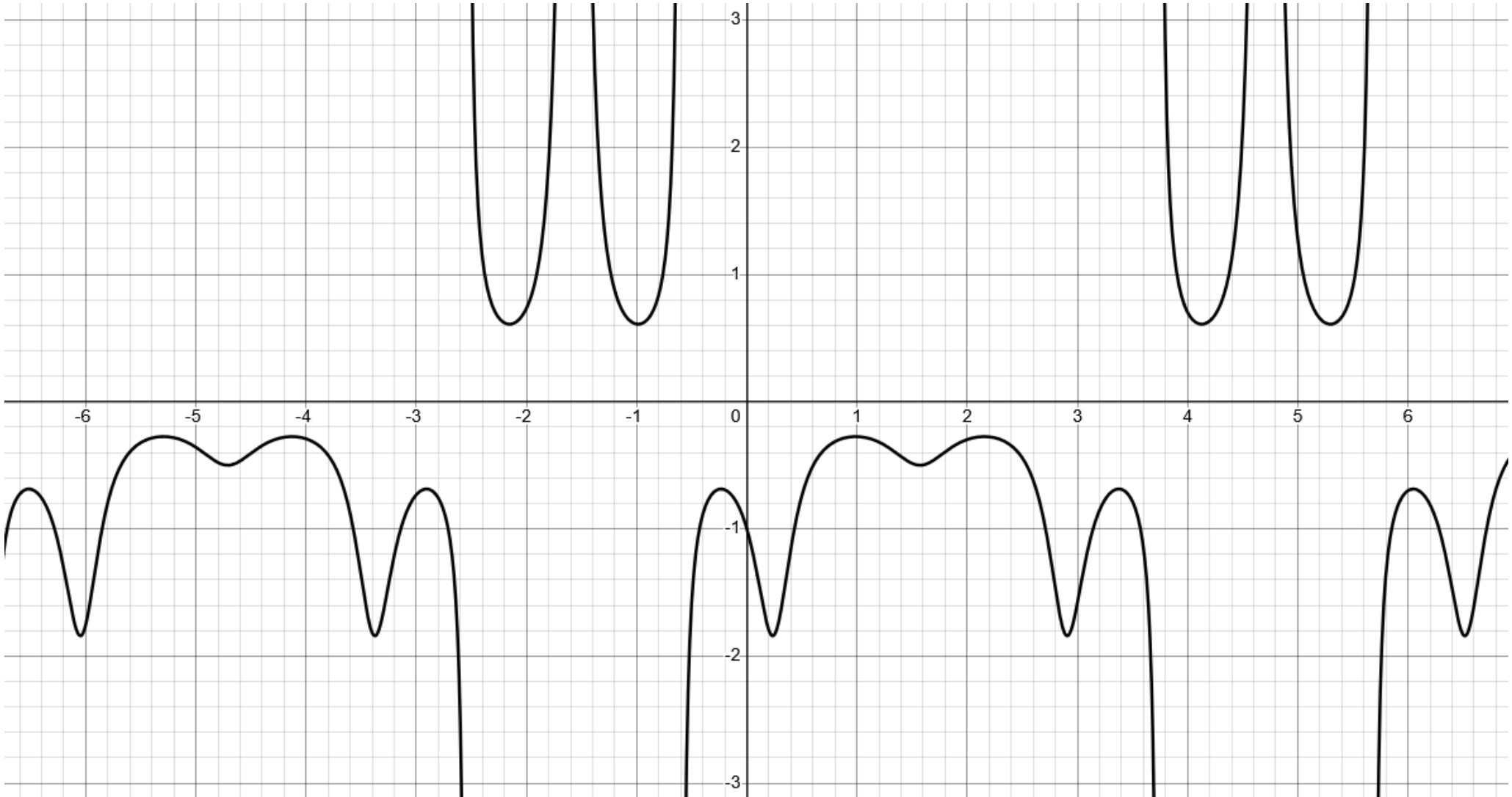
Ξ - comparison to not duplicate singular points

Δ - exclusion area around singular points

β - areas size around singular point in searching algorithm

Intel Core i5-12500H
Cores: 4 + 8 x 2.5 GHz + 1.8 GHz

$$y = \frac{1}{sin5x\ - 2sinx\ - 1}$$



Ξ 1e-6

Δ 1e-1

β 1e-2

$$\int_{-1,23}^{4,35} \frac{1}{sin5x - 2sinx - 1} dx$$ I ~ -2,26665

Segments	1	16	50	100
I	-2,256549	-2,256577	-2,256583	-2,25658
σ	0,000033034	0,0000053955	0,0000021167	0,0000005161
time	~ 0 sec	~ 0 sec	~ 0 sec	~ 0 sec

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Δ 1e-1

β 1e-2

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Δ 1e-2

β 1e-3

Segments	1	16	50	100
I	-2,265631	-2,26568	-2,265621	-2,265653
σ	0,0001369099	0,0000408182	0,0000249604	0,0000031594
time	~ 0 sec	~ 0 sec	~ 0 sec	~ 0 sec

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β 1e-2

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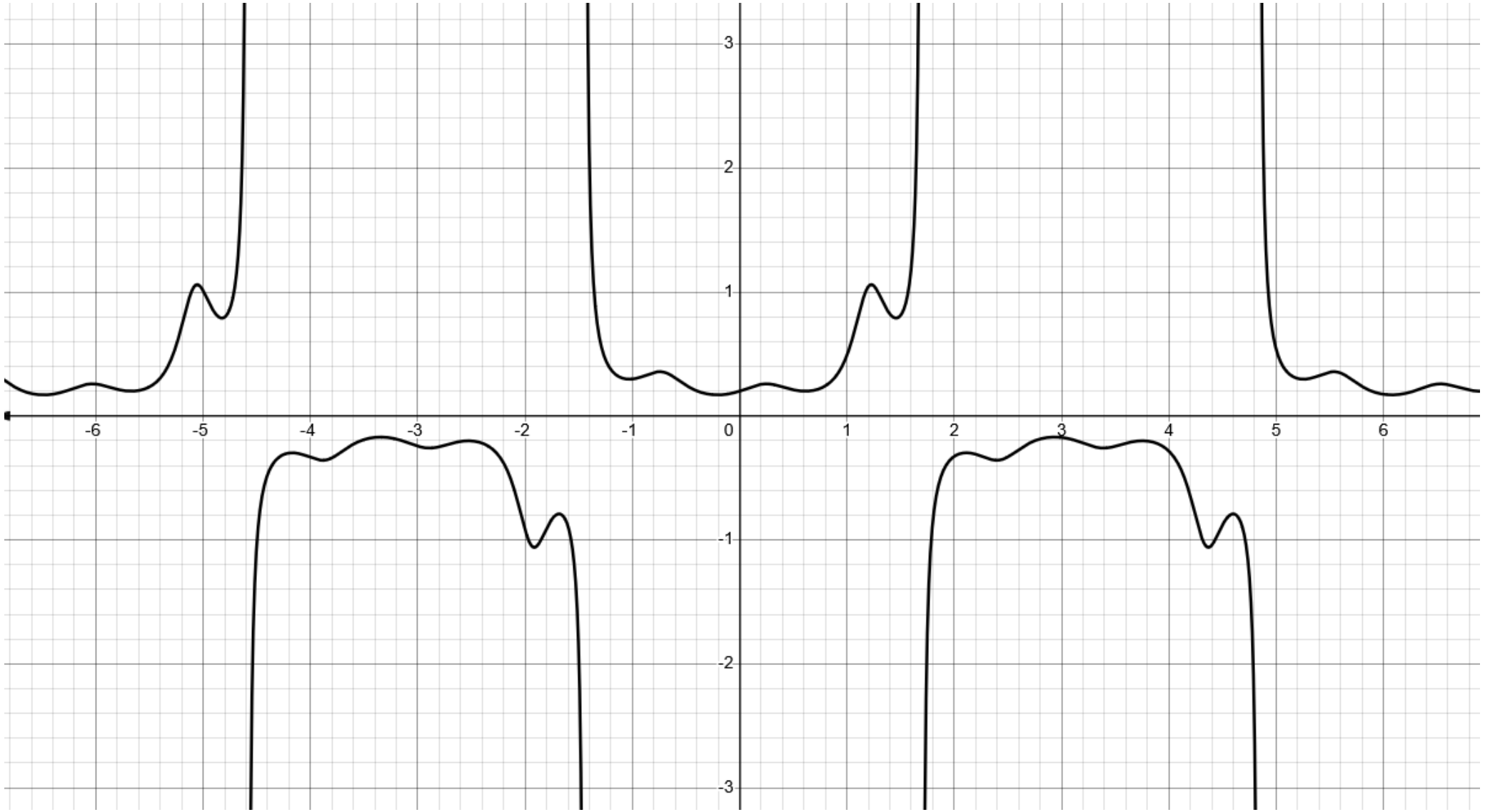
Δ 1e-3

β 1e-4

Segments	1	16	50	100
I	-2,266654	-2,266549	-2,26668	-2,2666731
σ	0,000233	0,00023186	0,0000834538	0,0001376972
time	~ 0 sec	~ 0 sec	~ 0 sec	~ 0 sec

$$y = \frac{1}{5cosx\ - sin7x}\,dx$$



Ξ 1e-6

Δ 1e-1

β 1e-2

$$\int_1^4 \frac{1}{5cosx - sin7x}\,dx$$

I ~ -0,0445533

Segments	1	16	50	100
I	-0,07394	-0,073305	-0,073306	-0,073306
σ	0,000019931	0,0000025002	0,000008323	0,0000002931
time	~ 0 sec	~ 0 sec	~ 0 sec	~ 0 sec

Δ 1e-2

β 1e-3

Segments	1	16	50	100
I	-0,047357	-0,047456	-0,047361	-0,047369
σ	0,0000630625	0,000021569	0,0000048429	0,0000023175
time	~ 0 sec	~ 0 sec	~ 0 sec	~ 0 sec

Δ 1e-3

β 1e-4

Segments	1	16	50	100
I	-0,044783	-0,044649	-0,044785	-0,044792
σ	0,00001012536	0,0000998288	0,0000447832	0,0000284539
time	~ 0 sec	~ 0 sec	~ 0 sec	~ 0 sec

Ξ 1e-6

Δ 1e-1

β 1e-2

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Ξ 1e-6

$$\int_1^4 \frac{1}{5cosx - sin7x}\,dx$$

I ~ -0,0445533

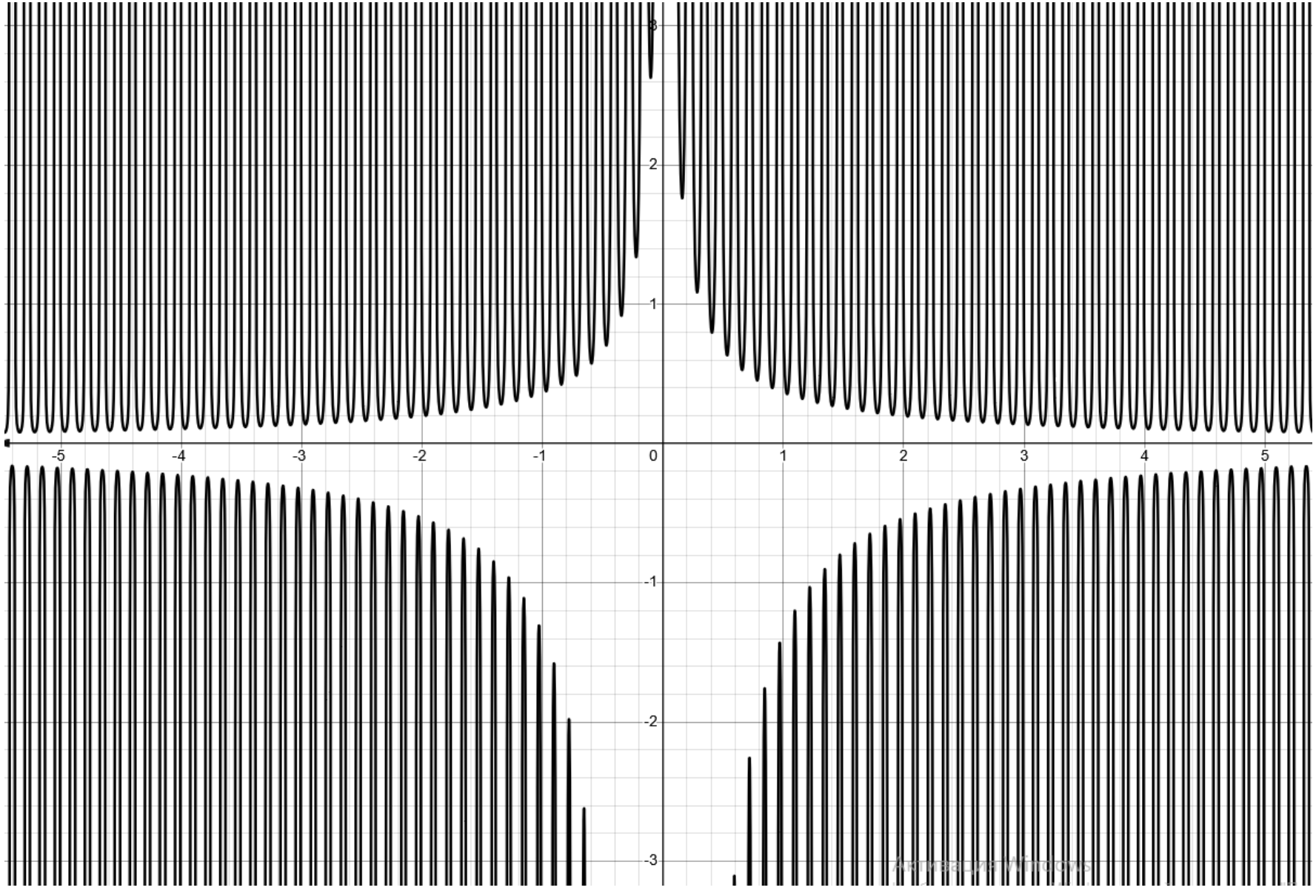
Δ 1e-3
β 1e-4

Segments	1	16	50	100
I	-0,044783	-0,044649	-0,044785	-0,044792
σ	0,00001012536	0,0000998288	0,0000447832	0,0000284539
time	~ 0 sec	~ 0 sec	~ 0 sec	~ 0 sec

Δ 1e-4
β 1e-5

Segments	1	16	50	100
I	-0,044325	-0,044275	-0,044579	-0,044545
σ	0,0003454395	0,0001367338	0,000044015	0,0000365304
time	~ 0 sec	~ 0 sec	~ 0 sec	~ 0 sec

$$y = \frac{1}{x^{\frac{2}{3}} + \sqrt{3x^2}\sin(16\pi x)}$$



Ξ **1e-6**

$$\int_{0,001}^{5} \frac{1}{x^{\frac{2}{3}} + \sqrt{3x^2}\sin(16\pi x)}\,dx \qquad I \sim 10{,}4727$$

1e-6

$$\int_{0,001}^{5} \frac{1}{x^{\frac{2}{3}} + \sqrt{3x^2}\sin(16\pi x)}\,dx$$

I ~ 10,4727 + (-0,53982)

1e-6

$$\int_{0,001}^{5} \frac{1}{x^{\frac{2}{3}} + \sqrt{3x^2}\sin(16\pi x)}\,dx$$

I ~ 9,93045

Ξ 1e-6

$$\int_{0,001}^{5} \frac{1}{x^{\frac{2}{3}} + \sqrt{3x^2}\sin(16\pi x)}\,dx \qquad I \sim 9{,}93045$$

Δ 1e-1
β 1e-2

Segments	1	100
I	0,175901	-0,649805
σ	0,045552269	0,7338817461
time	~ 0 sec	~ 0 sec

Δ 1e-2
β 1e-3

Segments	1	100
I	10,511931	8,839051
σ	1,034853021	0,7378814038
time	~ 0 sec	~ 0 sec

Δ 1e-3
β 1e-4

Segments	1	100
I	9,89919	8,560443
σ	0,20892145	0,74085204
time	~ 0 sec	~ 0 sec

Δ 1e-4
β 1e-5

Segments	1	100
I	10,647619	8,611243
σ	0,631374422	0,737240684
time	~ 0 sec	~ 0 sec

Δ 1e-5
β 1e-6

Segments	1	100
I	8,327777	8,584991
σ	1,739279487	0,988177556
time	~ 0 sec	~ 0 sec

Ξ 1e-6

$$\iint_{-1}^{1} \frac{1}{sinx - cosy} dxdy$$

I ~ 6,92

Δ 1e-1
β 1e-2

Segments	1	100
I	-5,99032	-5,98887
σ	0,00312502	0,001274428
time	~ sec	~ 1 sec
singularities	172	172

Δ 1e-2
β 1e-3

Segments	1	100
I	-6,894758	-6,89198
σ	0,01086249	0,0015001304
time	~ 19 sec	~ 2 sec
singularities	1720	1720

Δ 1e-3
β 1e-4

Segments	1	100
I	-6,980284	-6,996705
σ	0,0213981	0,0067015817
time	~ 213 sec	~ 23 sec
singularities	17172	17172

Δ 1e-4
β 1e-5

Segments	1	100
I	-7,999879	158,077692
σ	3,4079284	162,27748148
time	~ 608 sec	~ 103 sec
singularities	42930	42930

Calculation of $\pi^0 \to \gamma\gamma$

The two-photon decay coupling constant

$$g_{\pi\gamma\gamma} = G_{\pi\gamma\gamma}(M_\pi^2, 0, 0) \simeq \frac{mN_\pi}{4\pi^2\Lambda_\pi^2} I(M_\pi^2, 0, 0),$$

the decay width

$$\boxed{\Gamma(\pi^0 \to \gamma\gamma) = \frac{M_\pi^3}{64\pi}(4\pi\alpha)^2 g_{\pi\gamma\gamma}^2} \qquad (7)$$

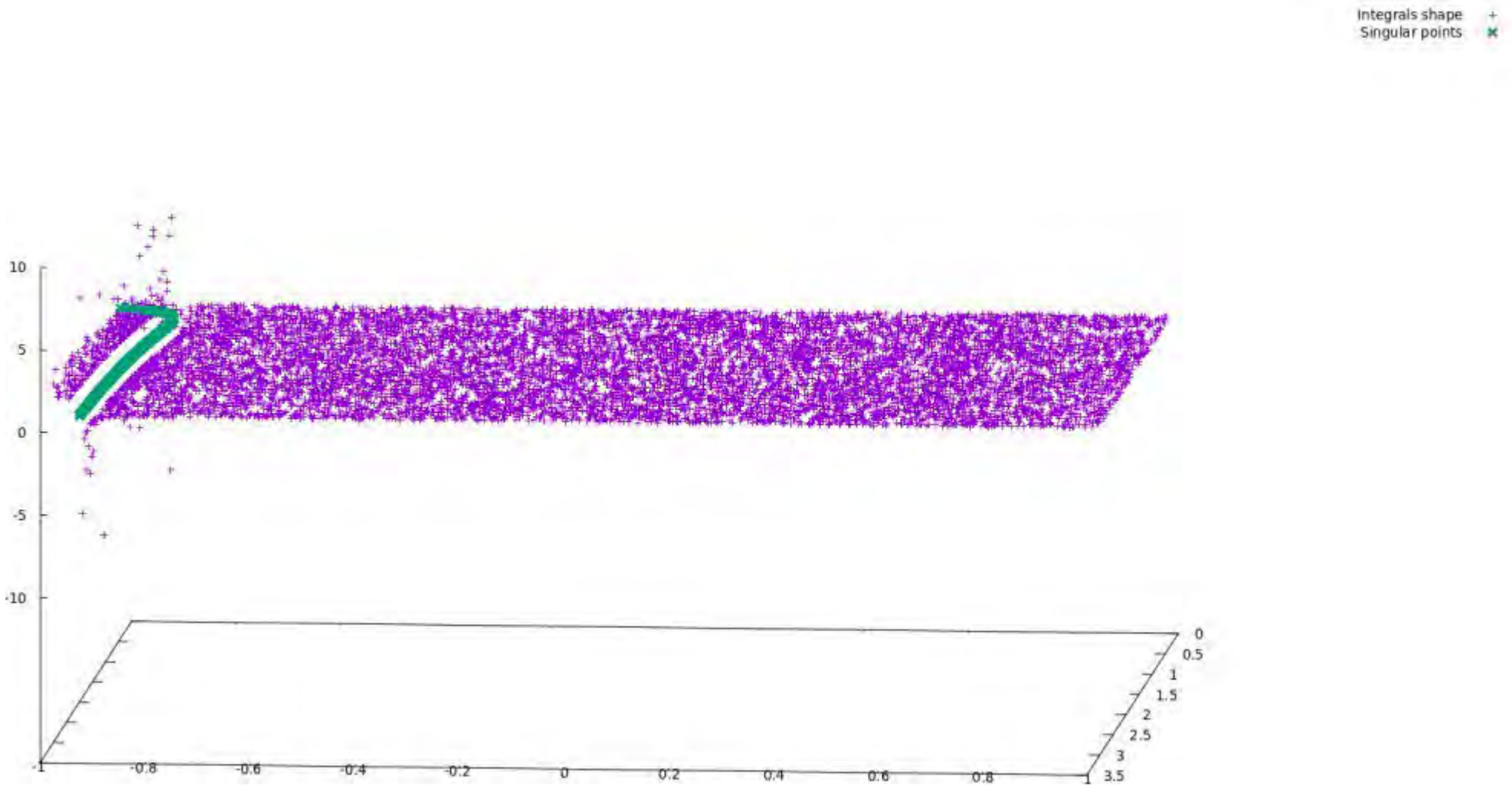
The transition form factor

$$\boxed{F_{\pi\gamma}(Q^2) = e^2 G_{\pi\gamma\gamma}(M_\pi^2, Q^2, 0)} \qquad (8)$$

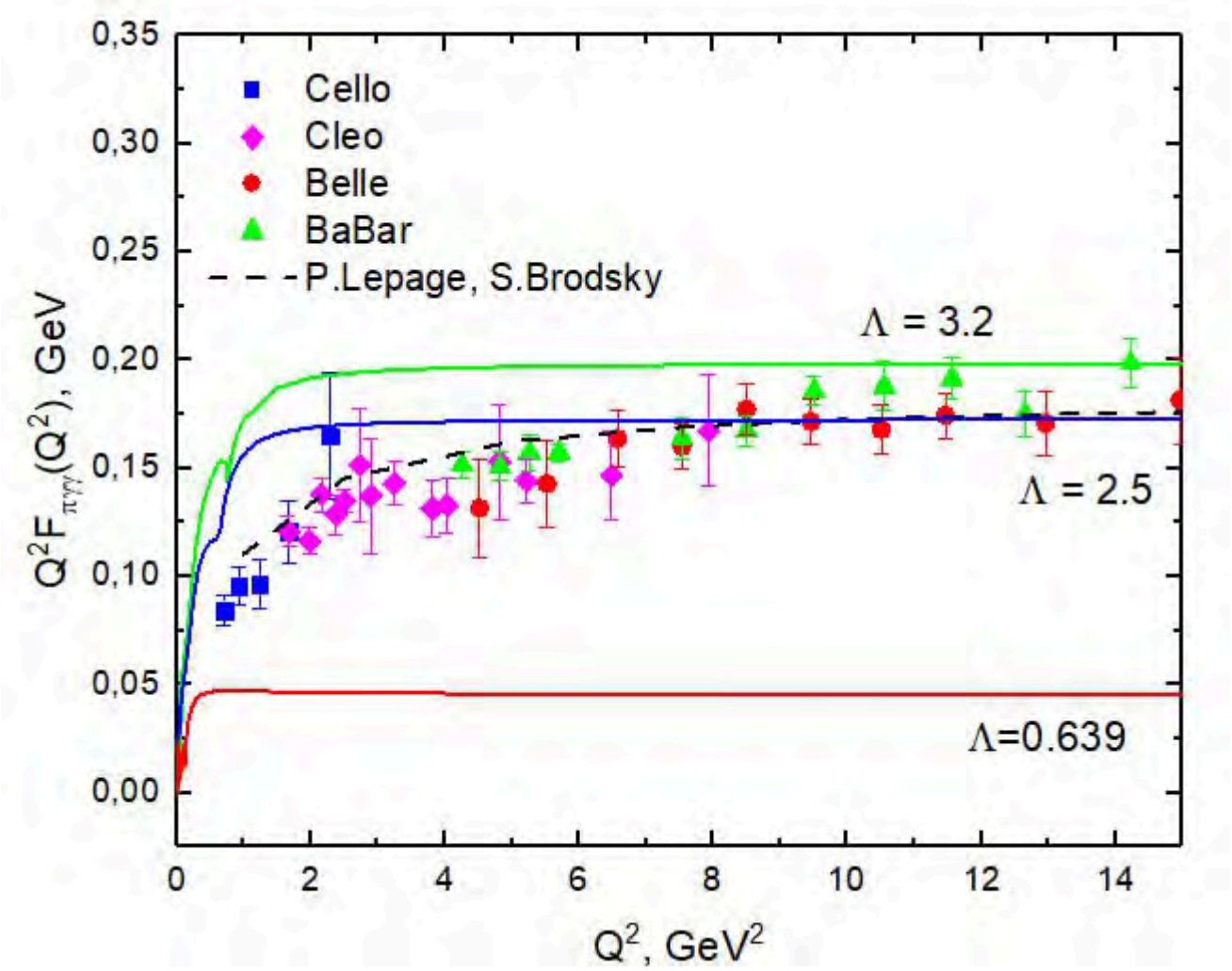
Calculation of $\pi^0 \to \gamma\gamma$

$$ReI_3 = \frac{1}{8\pi^2}\int_m^{\sqrt{\Lambda^2+m^2}} \sqrt{E^2-m^2}\int_{-1}^{1} d\cos\theta \left\{ f(E-\mu)\left[-\frac{1}{2Eq_{10}-2\bar{p}q_{10}\cos\theta}\frac{1}{2EP_0+q_2^2} \right.\right.$$
$$\left.\left. -\frac{1}{2Eq_{20}+2\bar{p}q_{10}\cos\theta}\frac{1}{2EP_0+q_2^2} + \frac{1}{2Eq_{10}-2\bar{p}q_{10}\cos\theta}\frac{1}{2EP_0+P_0^2} \right.\right.$$

Calculation of $\pi^0 \rightarrow \gamma\gamma$



Calculation of $\pi^0 \rightarrow \gamma\gamma$



Summary

- Algorithm works and gives quite accurate results, but one needs to understand the nature of the problem and select the algorithm's parameters
- Parallelization allows to significantly reduce the calculation time
- The algorithm was applied to calculate $\pi_0 \to \gamma\gamma$ problem
- The algorithm will be applied for further calculations of integrals with an unknown singularity