Geometry-dependent classicality of qutrits on low-dimensional orbits

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The XXIV International Scientific Conference of Young Scientists and Specialists (AYSS-2020)

Wigner quasiprobability distribution: $W(\Omega_N) = \mathrm{tr}\left[\varrho\, \Delta(\Omega_N)\right],$

- density matrix $\varrho \in \mathfrak{P}_N$: $\varrho = \varrho^\dagger, \quad \varrho \geq 0, \quad \mathrm{tr}(\varrho) = 1,$
- Stratonovich-Weyl kernel $\Delta(\Omega_N) \in \mathfrak{P}^*_N$: $\Delta = \Delta^\dagger, \ \mathrm{tr}(\Delta) = 1, \ \mathrm{tr}\left(\Delta^2\right) = N,$
- phase space $\Omega_N \to \mathbb{F}^N_{d_1, d_2, \dots, d_s} = U(N)/H$ is a complex flag manifold with the isotropy group $H = U(k_1) \times U(k_2) \times \cdots \times U(k_{s+1}) \in U(N)$, where $k_1 = d_1, \ k_{i+1} = d_{i+1} - d_i, \quad d_{s+1} = N, \quad d_i \in \mathbb{Z}^+ .$

Global indicator of classicality:

$$\mathcal{Q}_N = \frac{\text{Volume of orbit subspace } \mathcal{O}[\mathfrak{P}^{(+)}_N]}{\text{Volume of orbit space } \mathcal{O}[\mathfrak{P}_N]},$$

where $\mathcal{O}[\mathfrak{P}^{(+)}_N]$ is the unitary orbit space of states with non-negative WF.

Introduction

Density matrix in the Bloch form:

$$\varrho_\xi = \frac{1}{N}\left(I + \sqrt{\frac{N\,(N-1)}{2}}\,(\boldsymbol{\xi}, \boldsymbol{\lambda})\right),$$

- $\boldsymbol{\xi}$ is (N^2-1)-dimensional Bloch vector,
- $\boldsymbol{\lambda} = \{\lambda_1\,, \cdots, \lambda_{N^2-1}\}$ is $\mathfrak{su}(N)$ algebra orthonormal Hermitian basis.

Stratonovich-Weyl kernel:

$$\Delta(\Omega|\boldsymbol{\nu}) = \frac{1}{N} U(\Omega)\left(I + \kappa \sum_{\lambda_s \in K} \mu_s(\boldsymbol{\nu})\lambda_s\right) U(\Omega)^\dagger\,, \quad \kappa = \sqrt{N(N^2-1)/2}\,,$$

- $K \in \mathfrak{su}(N)$ is Cartan subalgebra,
- real coefficients $\mu_3^2 + \mu_8^2 + \cdots + \mu_{s^2-1}^2 = 1\,, \quad s = \overline{2, N}\,,$
- parameter $\boldsymbol{\nu} = (\nu_1\,, \cdots, \nu_{N-2})$ labels members of the WF family.

Family of the Wigner functions:

$$W^{(\nu)}_{\boldsymbol{\xi}}(\Omega_N) = \frac{1}{N}\left(1 + \frac{N^2-1}{\sqrt{N+1}}(\boldsymbol{n}, \boldsymbol{\xi})\right),$$

- vector $\boldsymbol{n} = \mu_3 \boldsymbol{n}^{(3)} + \mu_8 \boldsymbol{n}^{(8)} + \cdots + \mu_{N^2-1}\boldsymbol{n}^{(N^2-1)}$,
- orthonormal vectors $\boldsymbol{n}^{(s^2-1)}_{\mu} = \frac{1}{2}\,\mathrm{tr}\left(U\lambda_{s^2-1}U^{\dagger}\lambda_{\mu}\right)$.

Wigner function is non-negative, $W^{(\nu)}_{\boldsymbol{\xi}}(\Omega_N) \geq 0$, for any state with

$$0 \leq \boldsymbol{\xi}^2 \leq r^2_{*}(N)\,, \qquad r_{*}(N) = \sqrt{N+1}/(N^2-1)\,.$$

Definition 1.

The unitary orbit space $\mathcal{O}[\mathfrak{P}_N]$ is the quotient space under the equivalence relation imposed by the adjoint $SU(N)$ action on the state space $\mathfrak{P}_N$ with quotient (canonical) mapping $\pi : \mathfrak{P}_N \longrightarrow \mathcal{O}[\mathfrak{P}_N] = \mathfrak{P}_N/SU(N)$.

Definition 2.

The set $\Omega_N^{(+)}[\varrho] = \{ x \in \Omega_N \mid W_\varrho(\Omega_N) \geq 0 \}$ is a subset of phase space Ω_N where the Wigner function of a given state ϱ is non-negative.

Definition 3.

The subspace $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ is composed from states $\varrho \in \mathfrak{P}_N : \Omega_N^{(+)}[\varrho] = \Omega_N$.

Definition 4.

The subset $\mathcal{O}[\mathfrak{P}_N^{(+)}] = \pi[\mathfrak{P}_N^{(+)}] = \{\pi(x) \mid x \in \mathfrak{P}_N^{(+)} \}$ represents the image of $\mathfrak{P}_N^{(+)}$ under the quotient mapping π.

The orbit space $\mathcal{O}[\mathfrak{P}_N]$ can be realised as an ordered $(N-1)$-simplex in the space of eigenvalues $\boldsymbol{r} = \{r_1, r_2, \dots, r_N\}$ of a density matrix $\varrho = U \varrho_{diag} U^\dagger$:

$$\mathcal{C}^{(N-1)} = \left\{ \boldsymbol{r} \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^{N} r_i = 1, \quad 1 \geq r_1 \geq r_2 \geq \cdots \geq r_{N-1} \geq r_N \geq 0 \right\}.$$

The subspace $\mathcal{O}[\mathfrak{P}_N^{(+)}]$ is a dual cone $(\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) = r_1\pi_N + r_2\pi_{N-1} + \cdots + r_N\pi_1$ of a subset $\mathcal{O}[\mathfrak{P}_N] \subset \mathbb{R}^{N-1}$:

$$\mathcal{O}[\mathfrak{P}_N^{(+)}] = \left\{ \boldsymbol{\pi} \in \mathbf{spec}\,(\Delta(\Omega_N)) \;\mid\; (\boldsymbol{r}^\downarrow, \boldsymbol{\pi}^\uparrow) \geq 0, \quad \forall\, \boldsymbol{r} \in \mathcal{O}[\mathfrak{P}_N] \right\},$$

where $\boldsymbol{\pi} = \{\pi_1, \pi_2, \dots, \pi_N\}$ are the eigenvalues of the Stratonovich-Weyl kernel $\Delta(\Omega_N \,|\, \boldsymbol{\nu})$.

Information and Riemannian metrics

Monotone metric

Let

- M_n be the set of all complex $n \times n$ matrices;
- $\mathcal{M} \subset M_n$ be the manifold of all complex positive-definite $n \times n$ matrices;
- $\mathcal{D} \subset \mathcal{M}$, $\mathcal{D} = \{\rho \in \mathcal{M} \,:\, Tr(\rho) = 1\}$, be the manifold of all density matrices;
- the tangent space $\mathcal{T}_\rho(\mathcal{M}) = \{x \in M_n \,:\, x = x^*\}$ of $\mathcal{M}$ at $\rho \in \mathcal{M}$ be the set of all $n \times n$ Hermitian matrices;
- the tangent space $\mathcal{T}_\rho(\mathcal{D})$ at $\rho \in \mathcal{D}$ be the subspace of traceless matrices in $\mathcal{T}_\rho(\mathcal{M})$.

A Riemannian metric λ on $\mathcal{M}$ is called **monotone metric** if the inequality

$$\lambda_{h(\rho)}(h(u), h(u)) \leq \lambda_\rho(u, u)$$

holds for any $\rho \in \mathcal{M}$, $u \in \mathcal{T}_\rho(\mathcal{M})$ and any completely positive trace preserving mapping h, called **stochastic mapping**.

In fact, λ is monotone iff it can be written as

$$\lambda_\rho(u,v) = Tr\left(uJ_\rho(v)\right)\,, \qquad J_\rho = \frac{1}{f(L_\rho/R_\rho)R_\rho}\,,$$

where $L_\rho\,, R_\rho$ are the left and right multiplication operators, and $f\,:\,(0,\infty) \longrightarrow \mathbb{R}$ is a symmetric, $f(t) = t\,f(t^{-1})$, and normalized, $f(1) = 1$, operator monotone function.

If v and ρ commute, then $J_\rho(v) = \rho^{-1}v$, and so any monotone metric is equal to the Fisher information metric on commutative submanifolds.

Therefore, monotone metrics generalize the Fisher information metric on the class of probability densities (classical or commutative case) to the class of density matrices (quantum or noncommutative case) which are used in Quantum Statistics and Information Theory.

A monotone metric $\lambda_\rho(u,v) = Tr\left(u \frac{1}{f(L_\rho/R_\rho)R_\rho}(v)\right)$ can be rewritten as $\lambda_\rho(u,v) = Tr\left(u\, c(L_\rho, R_\rho)(v)\right)$, where $c(x,y) = \frac{1}{f(x/y)y}$ is the related to λ **Morozova-Chentsov function**.

For the **Bures**, **Bogoliubov-Kubo-Mori**, **Wigner-Yanase-Dyson** metrics, correspondingly, the operator monotone functions are

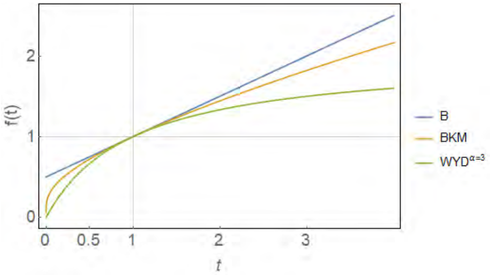
$$f_B(t) = \frac{1+t}{2}, \quad f_{BKM}(t) = \frac{t-1}{\ln t}, \quad f_{WYD}(t;\alpha) = \frac{1-\alpha^2}{4}\frac{(t-1)^2}{(t^{\frac{1-\alpha}{2}}-1)(t^{\frac{1+\alpha}{2}}-1)},$$

for the Morozova-Chentsov functions $c_B(x,y) = \frac{2}{x+y}$, $c_{BKM}(x,y) = \frac{\ln x - \ln y}{x-y}$, $c_{WYD}(x,y;\alpha) = \frac{4}{1-\alpha^2}\frac{(x^{\frac{1-\alpha}{2}} - y^{\frac{1-\alpha}{2}})(x^{\frac{1+\alpha}{2}} - y^{\frac{1+\alpha}{2}})}{(x-y)^2}$, where $\alpha \in [-3,3]$.

For $\alpha = \pm 1$ one obtains the Bogoliubov-Kubo-Mori metric, and for $\alpha = \pm 3$ - the right logarithmic derivative metric, which is the greatest monotone metric with $f(t) = \frac{2t}{1+t}$ and $c(x,y) = \frac{x+y}{2xy}$.

Morozova-Chentsov functions f(t)

The maximal function for the Bures (B) metric is a straight line, while the minimal function for the Wigner-Yanase-Dyson (WYD) metric for $\alpha = \pm 3$ is a hyperbola. The intermediate case leads to the Bogoliubov-Kubo-Mori (BKM) metric used in quantum statistical mechanics.



These metrics are Fisher-adjusted, $f(1) = 1$. The Bures metric is also Fubini-Study adjusted, $f(0) = 1/2$.

The respective metrics are:

$$g_f = \frac{1}{4}\sum_{i=1}^{N}\frac{\mathrm{d}r_i\otimes\mathrm{d}r_i}{r_i} + \frac{1}{2}\sum_{i<j}^{N} c_f(r_i\,,r_j)(r_i-r_j)^2\,\left(U^\dagger\mathrm{d}U\right)_{ij}\otimes\left(U^\dagger\mathrm{d}U\right)_{ij}\,.$$

The **global indicator of classicality** $\mathcal{Q}_N$ of N-dimensional quantum system is defined as the following ratio:

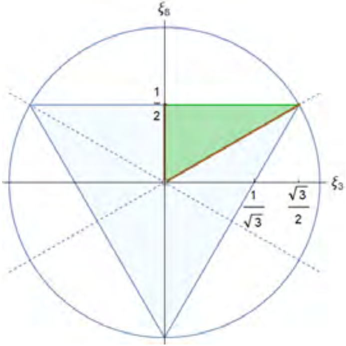
$$\mathcal{Q}_N[\mathrm{g_f}] = \frac{\int\cdots\int_{\mathcal{O}[\mathfrak{P}_N^{(+)}]}\mathrm{dP}_N(\mathrm{g}|\boldsymbol{r})}{\int\cdots\int_{\mathcal{O}[\mathfrak{P}_N]}\mathrm{dP}_N(\mathrm{g}|\boldsymbol{r})}\,,$$

where the state space $\mathfrak{P}_N$ is endowed with a Riemannian metric g, and $\mathrm{dP}_N(\mathrm{g}|\boldsymbol{r}) = \sqrt{\det||\mathrm{g}||}\,\mathrm{d}r_1\wedge\mathrm{d}r_2\wedge\cdots\wedge\mathrm{d}r_N$ is a corresponding measure.

Qutrit

Qutrit density matrix:

$$\varrho_3 = \frac{1}{3}(I + \sqrt{3}\sum_{\nu=1}^{8} \xi_\nu \lambda_\nu)\,.$$



Density matrix spectrum $\{r_1, r_2, r_3\}$:

$$1 \geq r_1 \geq r_2 \geq r_3 \geq 0\,.$$

Ordered simplex $\overline{\Delta}_2$:

$0 \leq \xi_3 \leq \sqrt{3}/2\,, \quad \xi_3/\sqrt{3} \leq \xi_8 \leq 1/2\,.$

Strata of a qutrit orbit space:

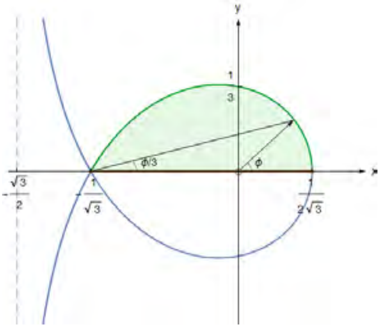
- stratum $\mathcal{O}_{123}$ of $\dim(\mathcal{O}) = 6$ corresponding to regular orbits;
- two strata $\mathcal{O}_{1|23}$ and $\mathcal{O}_{12|3}$ with $\dim(\mathcal{O}_{1|23}) = \dim(\mathcal{O}_{12|3}) = 4\,;$
- stratum $\mathcal{O}_0$ of $\dim(\mathcal{O}_0) = 0$ corresponding to the orbit of maximally mixed state.

Under transformation

$$\xi_3 = \sqrt{3} r \sin\left(\frac{\varphi}{3}\right), \ \xi_8 = \sqrt{3} r \cos\left(\frac{\varphi}{3}\right)$$

the simplex $\overline{\Delta}_2 \longrightarrow \mathcal{O}(\mathfrak{P}_3)$:

$$\left\{ r \geq 0\,, \varphi \in [0, \pi] \,\middle|\, \cos\left(\frac{\varphi}{3}\right) \leq \frac{1}{2\sqrt{3}r} \right\}.$$



Maclaurin trisectrix in polar coordinates $(x = r\cos\varphi\,, y = r\sin\varphi)$:

$$r(\varphi, 1/\sqrt{3}) = \frac{1}{2\sqrt{3}\cos(\varphi/3)}.$$

Qutrit Stratonovich-Weyl kernel: $\Delta = U\,\frac{1}{3}(I + 2\sqrt{3}(\mu_3\lambda_3 + \mu_8\lambda_8))\,U^\dagger\,.$

SW kernel spectrum $\{\pi_1, \pi_2, \pi_3\}$:

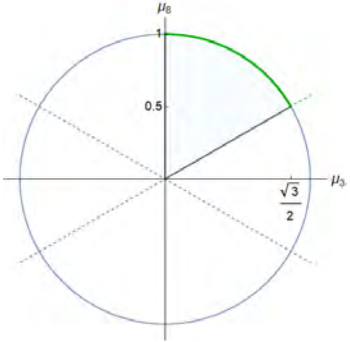
$$\pi_1 \geq \pi_2 \geq \pi_3\,.$$

Qutrit moduli space, $\mu_3^2 + \mu_8^2 = 1$:

$$\mu_3(\nu) = \frac{\sqrt{3}}{4}\sqrt{(1+\nu)(5-3\nu)}\,,$$

$$\mu_8(\nu) = \frac{1}{4}(1-3\nu)\,,\ \nu = \frac{1}{3} - \frac{4}{3}\cos(\zeta)\,,$$

so $\mu_3 = \sin\zeta\,,\ \mu_8 = \cos\zeta\,,\ \zeta \in [0, \frac{\pi}{3}]\,.$



Euler representation for elements of $SU(3)$ group, $\mathrm{Vol}[SU(3)] = \sqrt{3}\pi^5$:

$$U = e^{\imath\frac{\alpha}{2}\lambda_3} e^{\imath\frac{\beta}{2}\lambda_2} e^{\imath\frac{\gamma}{2}\lambda_3} e^{\imath\theta\lambda_5} e^{\imath\frac{a}{2}\lambda_3} e^{\imath\frac{b}{2}\lambda_2} e^{\imath\frac{c}{2}\lambda_3} e^{\imath\phi\lambda_8}\,,$$

where $\alpha, a \in [0, 2\pi]\,,\ \beta, b \in [0, \pi]\,,\ \gamma, c \in [0, 4\pi]\,,\ \theta \in [0, \pi/2]\,,\ \phi \in [0, \sqrt{3}\pi]\,.$

Wigner function is bounded, $W_N^{(-)} \leq W(\Omega_N) \leq W_N^{(+)}$, so that:

$$W_N^{(-)} = \sum_{i=1}^{N} \pi_i r_{N-i+1} \,, \qquad W_N^{(+)} = \sum_{i=1}^{N} \pi_i r_i \,.$$

Qutrit upper bound: $W_3^{(+)} = \frac{1}{3} + \frac{4r}{\sqrt{3}} \cos\left(\zeta - \frac{\varphi}{3}\right) > 0$.

$\zeta = 0$ $\qquad$ $\zeta = \pi/6$ $\qquad$ $\zeta = \pi/3$

Qutrit lower bound: $W_3^{(-)} = \frac{1}{3} - \frac{4r}{\sqrt{3}} \cos\left(\zeta + \frac{\varphi}{3} - \frac{\pi}{3}\right)$ is

- positive for $r \in [0, \frac{1}{4\sqrt{3}}]$;
- negative for $r \in [\frac{1}{2\sqrt{3}} \,, \frac{1}{\sqrt{3}}]$;
- negative only for certain values of ζ and φ for $r \in [\frac{1}{4\sqrt{3}} \,, \frac{1}{2\sqrt{3}}]$.

The Hilbert-Schmidt, Bures, Bogoliubov-Kubo-Mori and Wigner-Yanase-Dyson measures in the simplex of eigenvalues are:

$$\mathrm{dP}_3(\mathrm{g}_{\mathrm{HS}}|\boldsymbol{r}) = k_{\mathrm{HS}}\delta(\sum_{i=1}^{3} r_i - 1) \prod_{i<j}^{3} (r_i - r_j)^2 \,\mathrm{d}r_1 \wedge \mathrm{d}r_2 \wedge \mathrm{d}r_3 \,,$$

$$\mathrm{dP}_3(\mathrm{g}_{\mathrm{B}}|\boldsymbol{r}) = k_{\mathrm{B}}\delta(\sum_{i=1}^{3} r_i - 1) \frac{1}{\sqrt{r_1 r_2 r_3}} \prod_{i<j}^{3} \frac{(r_i - r_j)^2}{r_i + r_j} \,\mathrm{d}r_1 \wedge \mathrm{d}r_2 \wedge \mathrm{d}r_3 \,,$$

$$\mathrm{dP}_3(\mathrm{g}_{\mathrm{BKM}}|\boldsymbol{r}) = k_{\mathrm{BKM}}\delta(\sum_{i=1}^{3} r_i - 1) \frac{\prod_{i<j}^{3} (r_i - r_j)^2 \,\mathrm{d}r_1 \wedge \mathrm{d}r_2 \wedge \mathrm{d}r_3}{\sqrt{r_1 r_2 r_3}\, f_{BKM}(\frac{r_1}{r_2})\, f_{BKM}(\frac{r_2}{r_3}) f_{BKM}(\frac{r_3}{r_1})} \,,$$

$$\mathrm{dP}_3(\mathrm{g}_{\mathrm{WYD}}|\boldsymbol{r}) = k_{\mathrm{WYD}}\delta(\sum_{i=1}^{3} r_i - 1) \frac{\prod_{i<j}^{3} (r_i - r_j)^2 \,\mathrm{d}r_1 \wedge \mathrm{d}r_2 \wedge \mathrm{d}r_3}{\sqrt{r_1 r_2 r_3}\, f_{WYD}(\frac{r_1}{r_2})\, f_{WYD}(\frac{r_2}{r_3}) f_{WYD}(\frac{r_3}{r_1})} \,,$$

where k_f is a normalization constant.

Results

$\mathcal{Q}$**-indicators for low-dimensional orbits of a qutrit** are the following:

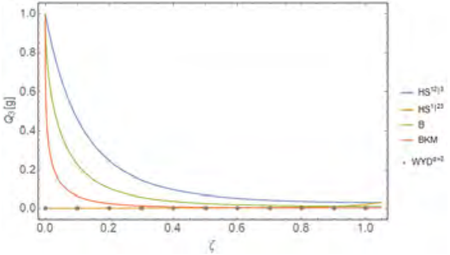
- $r_1 = r_2 \neq r_3$:

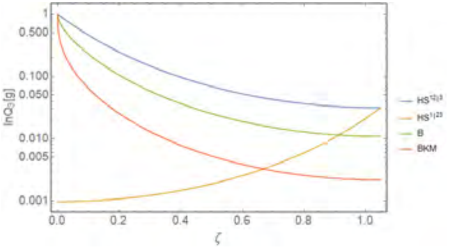
$$\mathcal{Q}_3^{12|3}[\mathrm{g_{HS}}] = \frac{1}{32}\csc^5\left(\frac{\pi}{6}+\zeta\right), \quad \mathcal{Q}_3[\mathrm{g_B}], \quad \mathcal{Q}_3[\mathrm{g_{BKM}}], \quad \mathcal{Q}_3[\mathrm{g_{WYD}}];$$

- $r_1 \neq r_2 = r_3$:

$$\mathcal{Q}_3^{1|23}[\mathrm{g_{HS}}] = \frac{\sec^5(\zeta)}{1024}, \qquad 0, \qquad 0, \qquad 0.$$

For the second type of orbits (which include pure states), the result for the mentioned monotone metrics is zero, since they mostly accumulate states close to pure ones, so the function under the integral has a simple pole at $(\xi_3, \xi_8) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ and the integral diverges.





The global $\mathcal{Q}$-indicator is sensitive to two features of classicality: the negativity of the Wigner function and local quantum uncertainty which is accumulated in the form of geometric measure on the orbit space inherited from a physically motivated quantum information content.

It can be either the quantum Fisher information I_F, or the Wigner-Yanase skew-symmetric information I_{WY}, both being natural generalizations of the classical Fisher information: $I_{WY} \leq I_F \leq 2I_{WY}$.

The dependence of the global indicator of classicality on the assigned geometry of a quantum state space is demonstrated for the Hilbert-Schmidt, Bures and Bogoliubov-Kubo-Mori ensembles of qutrits for low-dimensional orbits.