On the dichotomy of “elementary vs. composite” in Stratonovich-Weyl correspondence for qudits

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Contents

Quantum world in classical wording

"How far the [quantum] phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms"

N. Bohr (1949)

A vivid realizations of this Bohr principle is the language of quasiprobability distributions extensively used in phase-space formulation of

Statistical Model of Quantum Mechanics

Statistical average and expectation value

- **Statistical averages** of a function $A(q,p)$ on phase space Ω using the Probability Distribution Function (PDF) $f(q,p)$:

$$\mathbb{E}(A) = \int \mathrm{d}\Omega\, A(q,p)\, f(q,p)\,, \quad \text{with} \quad \int \mathrm{d}\Omega\, f(q,p) = 1\,.$$

- **Expectation values** of a Hermitian operator $\hat{A}$ acting on the Hilbert space $\mathcal{H}$ in a state ϱ:

$$\mathbb{E}(\hat{A}) = \mathrm{Tr}(\hat{A}\varrho)$$

The Weyl correspondence: A mapping between a class of functions on the phase space $\mathbb{R}^{2n}$ and the set of operators on $L(\mathbb{R}^{2n})$

The Wigner function for a pure quantum state $\varrho = |\psi\rangle\langle\psi|$ with residing in a 2-dimensional phase space

$$W_{|\psi\rangle}(x,p) = \frac{1}{\pi\hbar}\int \mathrm{d}y \langle x+y|\varrho|x-y\rangle e^{-2\imath py/\hbar}\,.$$

Language of the quasiprobability distributions

In the contemporary views on quantum physics and quantum information science the probability distribution functions perhaps play a role of a most constructive tool.

Variety of quasiprobability distributions associated to Wigner

- The Wigner function (WF) and all its relatives:
  - WF over non-compact phase space /continuous;
  - WF of a finite level quantum system over compact phase space;
  - WF of finite-dimensional system with discrete phase-space;

The Stratonovich-Weyl correspondence

The Stratonovich–Weyl (SW) mapping generalizes the Weyl mapping to the phase spaces of other types than $\mathbb{R}^{2n}$

The basic ideas of the construction of the quasiprobability distributions:

1. The co-adjoint orbit Ω of a certain Lie group G of symmetries of a given physical system is identified with the phase space;
2. The operator associating each point of the orbit $(z \in \Omega)$ with a self-adjoint operator $\Delta(z)$ is needed.

$$z \in \Omega_N \quad \overset{?}{\longmapsto} \quad \Delta \in \mathfrak{P}^*_N$$

Answer: System of the Stratonovich-Weyl axioms

The Stratonovich-Weyl axioms

WF of a state ϱ is linear functional $W_\varrho(\Omega_N) = \mathrm{Tr}\,(\varrho\Delta(\Omega_N))$ given by the kernel $\Delta(\Omega)$ defined over a phase space Ω subject to the conditions:

I. Reconstruction; ρ can be build from WF as

$$\rho = \int_\Omega \mathrm{d}\Omega_N\, \Delta(\Omega) W_\rho(\Omega)\,.$$

II. Hermicity; $\Delta(\Omega) = \Delta(\Omega)^\dagger$

III. Finite Norm; The state norm is given by the integral of WF

$$\mathrm{Tr}\rho = \int_\Omega \mathrm{d}\Omega W_\rho(\Omega), \qquad \int_\Omega \mathrm{d}\Omega\, \Delta(\Omega) = 1$$

IV. Covariance: The mapping $\rho' = U(\alpha)\rho U^\dagger(\alpha)$ induces the symplectic transformation of coordinates $\boldsymbol{z} \in \Omega$,

$$\Delta'(\boldsymbol{z}) = \Delta(\boldsymbol{z}')\,, \quad \boldsymbol{z}' = T_\alpha \boldsymbol{z}\,, \quad T_\alpha \in \mathrm{Sp}(d_N)\,. \tag{1}$$

Wigner function of N-level quantum system

The pair $\{\varrho, \Delta(z)\}$ determine the Wigner function:

- The density matrix $\varrho \in \mathfrak{P}_N$,
- The SW kernel $\Delta(\Omega_N) \in \mathfrak{P}^*_N$,

$$W_\varrho(\Omega_N) = \mathrm{Tr}\left(\varrho\Delta(\Omega_N)\right) .$$

Quantum state ϱ from the semi-positive cone of $N \times N$ Hermitian matrices:

$$\mathfrak{P}_N = \{X \in M_N(\mathbb{C}) \,|\, X = X^\dagger \,, \mathrm{Tr}X = 1 \,, X \geq 0 \,.\}$$

Stratonovich-Weyl kernel $\Delta(\Omega_N)$ from the dual space $\mathfrak{P}^*_N$:

$$\mathfrak{P}^*_N = \{X \in M_N(\mathbb{C}) \,|\, X = X^\dagger \,, \mathrm{Tr}X = 1 \,, \mathrm{Tr}X^2 = N \,.\}$$

Symplectic manifold and moduli space of qudit

The symplectic space Ω_N is identified with the co-adjoint orbits of $SU(N)$:

$$\Omega_N\Big|_{\mathcal{H}_N} \to \frac{U(N)}{\mathrm{Iso}(\Delta)}\,.$$

where $\mathrm{Iso}(\Delta) \subset U(N)$ is an isotropy group of SW kernel.

$$\Delta(\Omega_N) = U(\Omega_N)\mathrm{diag}(\pi_1, \pi_2, \dots, \pi_N)U(\Omega_N)^\dagger\,,$$

At the same time the moduli space $\mathcal{P}_N$ of SW kernels is given by the "master equations" on the corresponding orbit space $\mathcal{O}[\mathfrak{P}^*_N/U(N)]\,,$

$$\mathcal{P}_N: \qquad \sum_{i=1}^{N} \pi_i = 1 \qquad \sum_{i=1}^{N} \pi_i^2 = N$$

What about the dichotomy "composite versus elementary" ?

- Is the dichotomy already encoded in SW axioms?
- If it is beyond the postulates how one can extend SW axioms?

Quantum Composite Systems – "made from or made of"

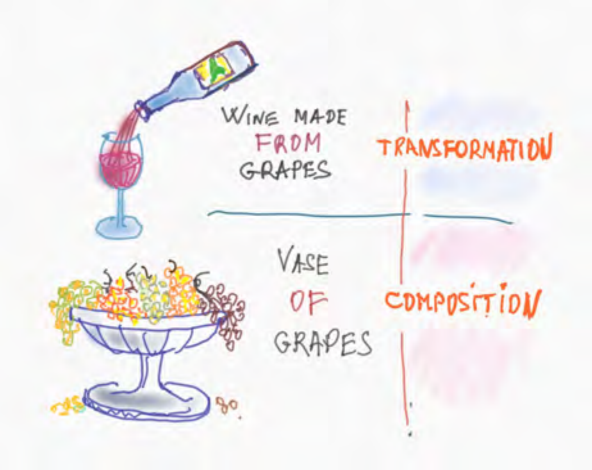

Figure 1: Metamorphosis vs. Formation

Elementary vs. Composite

Composite system postulate in Quantum Mechanics

The Hilbert space $\mathcal{H}_{AB}$ associated to a composite physical system AB is a subspace of the tensor product of the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_A$

$$\mathcal{H}_{AB} \subseteq \mathcal{H}_A \otimes \mathcal{H}_B \,,$$

corresponding to its components A and B.

Question:

But still, what is an elementary quantum system ?

Standpoint:

"It is theory finally which decides what can be observed as an elementary and what as a composite one ".

Elementary Quantum Systems

Irreducibility condition:

From the standpoint of symmetry an **elementary system** means its states change under the irreducible transformation of a certain physical symmetry

E.P. Wigner (1939)

Decomposition into elementary systems:

"Every system, even one consisting of an arbitrary number of particles, can be decomposed into elementary systems. **The usefulness of the decomposition into elementary** systems depends of how often one has deal with linear combinations containing several elementary systems. We consider a particle "elementary" if it does not appear to be useful to attribute structure to it"

T.T.Newton and E.P. Wigner (1949)

QM of composite systems

If n_A-dimensional system A and n_B-dimensional system B are joint together the Hilbert space of the resulting composite system is a subspace of the tensor product of the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of subsystems:

$$\mathcal{H}_{AB} \subset \mathcal{H}_A \otimes \mathcal{H}_B \,.$$

The state is given by the density matrix ϱ_{AB} acting on $\mathcal{H}_{AB}$, while an information on each subsystem is encoded in the density matrices ϱ_A and ϱ_B which are determined from ϱ_{AB} using the partial trace operation:

$$\varrho_A = \mathrm{tr}_B \varrho_{AB} \,, \quad \varrho_B = \mathrm{tr}_A \varrho_{AB} \,.$$

The partial trace operation is equivalent to the invariant integration over the unitary groups of subsystems,

$$\int_{U_B} \mathrm{d}\mu \, (I_A \otimes U_B) \, \varrho \, (I_A \otimes U_B^\dagger) = \varrho_A \otimes I_B \,.$$

Quasiprobability distributions of composite systems

Towards composition of quasiprobability distribution

1. Kolmogorov's $\sigma-$additivity axiom ;
2. Conditional probabilities ;
3. Compositions of SW kernels ;

A conventional assumption for SW kernels of a binary composite system:

$$\Delta_{AB} = \Delta_A \otimes \Delta_B \,,$$

Δ_A and Δ_B - the partially reduced SW kernels,

$$\Delta_A = \mathrm{tr}_B \Delta_{AB} \,, \quad \Delta_B = \mathrm{tr}_A \Delta_{AB} \,.$$

Experiment I: Searching for a "potential qubit pair" in quatrit

1. Create random ensemble of quatrit SW kernels;
2. Compute partially reduce matrices

$$\Delta_A = \mathrm{tr}_B \Delta_{AB}\,, \quad \Delta_B = \mathrm{tr}_A \Delta_{AB}\,.$$

3. Find the probability distribution function of random variables:

$$t_A = \mathrm{tr}\left({\Delta_A}^2\right), \quad t_B = \mathrm{tr}\left({\Delta_B}^2\right)\,.$$

Random SW kernels of 4-level system

Algorithm to generate random SW kernel Δ_{AB} for quatrit:

(I). From the normally distributed 3-vector $\boldsymbol{v} = (x_1, x_2, x_3)$:

$$N(0,1): \qquad \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\, x_i^2\right), \quad i = 1, 2, 3\,.$$

generate the uniform distribution $\boldsymbol{\mu} = (\mu_3, \mu_6, \mu_{15}) = \boldsymbol{v}/\mid \boldsymbol{v} \mid$ on $\mathbb{S}_2$;

(II). Construct the spectrum of quatrit SW kernel

$$\begin{aligned} \pi_1 &= \frac{1}{4} + \frac{\sqrt{15}}{4}(\mu_3 + \mu_6 + \mu_{15})\,, \; \pi_2 = \frac{1}{4} + \frac{\sqrt{15}}{4}(\mu_3 - \mu_6 - \mu_{15})\,, \\ \pi_3 &= \frac{1}{4} + \frac{\sqrt{15}}{4}(-\mu_3 + \mu_6 - \mu_{15})\,, \; \pi_4 = \frac{1}{4} + \frac{\sqrt{15}}{4}(-\mu_3 - \mu_6 + \mu_{15}) \end{aligned}$$

(III). Generate 4×4 matrix U from the Haar unitary ensemble;

(IV) Construct the quatrit SW kernel, $\Delta_{AB} = U \mathrm{diag}(\pi_1, \pi_2, \pi_3, \pi_4) U^\dagger$

Distribution of the "potential qubit pair" in quatrit

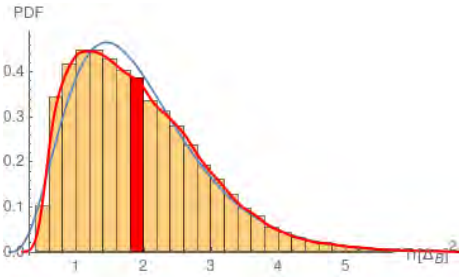

Figure 2: Probability distribution function of random variable $t_B = \mathrm{tr}(\Delta_B^2)$. The red line corresponds to the smooth kernel distribution which gives the expectation $\mathbb{E}[t_B] = 1.9234$. The blue line corresponds to the gamma distribution $\Gamma[4, 1/2]$ which gives the expectation value 2.

Distribution of the "potential qubit pair" in quatrit

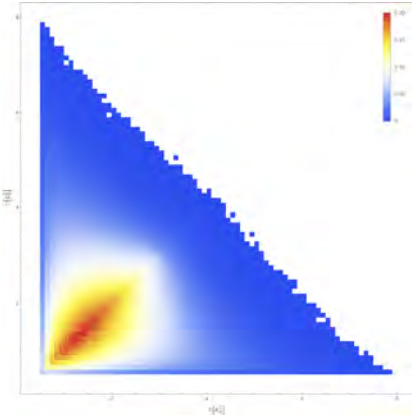

Figure 3: Probability distribution function of t_A, t_B .

Composite system postulate for SW kernel

The fifth SW postulate applicable to the case of composite systems.

V. Composite system postulate

The partially reduced matrices Δ_A and Δ_B are SW kernels of subsystems A and providing the SW mapping with the Wigner functions of the partially reduced states ϱ_A and ϱ_B respectively:

$$\begin{aligned} W_{\varrho_A} &= \mathrm{Tr}\left(\varrho_A \Delta_{N_A}\right) , \\ W_{\varrho_B} &= \mathrm{Tr}\left(\varrho_B \Delta_{N_B}\right) . \end{aligned}$$

This is equivalent to the following equations for SW kernel of joint system:

$$\mathrm{tr}_A\left(\mathrm{tr}_B \Delta(\Omega_N)\right)^2 = N_A , \qquad \mathrm{tr}_B\left(\mathrm{tr}_A \Delta(\Omega_N)\right)^2 = N_B .$$

Symmetry and Composition

Symmetry of an elementary and finite-dimensional system:

- The unitary symmetry $SU(N)$ of the Hilbert space $\mathbb{C}^N$ is the Global Unitary symmetry of an elementary $N-$level system;

Symmetry of a composite finite-dimensional system:

- If the Hilbert space structure is specified by the tensor Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$:

$$\mathcal{H}_{AB} \subset \mathcal{H}_A \otimes \mathcal{H}_B \,.$$

then the Local Unitary symmetry of the corresponding composite system is the subgroup $LU = SU(N_A) \times SU(N_B) \subset SU(N)$.

Symplectic manifold of a qudit

If a priory it is known that a quantum system is a composite, searching for the phase space $\Omega_{N_A \times N_B}$ we based on the correspondence:

$$\text{Global Unitary symmetry} \quad \Longleftrightarrow \quad \text{Local Unitary symmetry}$$

The symplectic manifold $\Omega_{N_A \times N_B}$ is defined as the LU group orbits

$$\Omega_N \Big|_{\mathcal{H}_A \otimes \mathcal{H}_B} \to \frac{U(N_A) \times U(N_B)}{H_X} \tag{2}$$

of element X from the moduli space $\mathcal{P}_{N_A \times N_B}$ of a composite SW kernel:

$$\mathcal{P}_{N_A \times N_B} = \left\{ X \in \mathfrak{P}^*_{N_A N_B} \,\middle|\, \mathrm{tr}_A \left(\mathrm{tr}_B X\right)^2 = N_A \,,\; \mathrm{tr}_B \left(\mathrm{tr}_A X\right)^2 = N_A \right\}$$

with a certain isotropy grou $H_X \subset U(N_A) \times U(N_B)$.

Comparing quatrit with 2-qubits

Our task is to compare 4-level system, the quatrit, and 2-qubit systems:

1. Describe the phase space Ω_4 vs. $\Omega_{2\times 2}$
2. Describe the moduli space $\mathcal{P}_4$ vs. $\mathcal{P}_{2\times 2}$

The basic guide is the symmetry

- For an elementary 4-level system $SU(4)$ is Global Unitary symmetry;
- For 2 qubits the Hilbert space is associated with the tensor product and the Local Unitary symmetry is the subgroup $K = SU(2) \times SU(2) \subset SU(4)$.

Quatrit: Phase-space

A generic **quatrit** ($N = 4$) state is given by the density matrix

$$\varrho_{\mathrm{Quatrit}} = \frac{1}{4}\left(\mathbb{I}_4 + \sqrt{6}\sum_{\alpha=1}^{15}\xi_\alpha\lambda_\alpha\right).$$

The generic **Stratonovich-Weyl kernel**

$$\Delta(\Omega_N) = \frac{1}{4}\mathbb{I} + \frac{\sqrt{15}}{4}\,U(\Omega_N)\,(\mu_3\lambda_3 + \mu_6\lambda_6 + \mu_{15}\lambda_{15})\,U(\Omega_N)^\dagger\,.$$

The **Wigner function** of a quatrit

$$W_{\boldsymbol{\xi}}(\Omega_4) = \frac{1}{4} + \frac{3\sqrt{5}}{4}\left[\mu_3(\boldsymbol{n}^{(3)},\boldsymbol{\xi}) + \mu_6(\boldsymbol{n}^{(6)},\boldsymbol{\xi}) + \mu_{15}(\boldsymbol{n}^{(15)},\boldsymbol{\xi})\right],$$

with

$$n_\alpha^{(s)} = \frac{1}{2}\mathrm{Tr}\left[U\lambda_s U^\dagger\lambda_\alpha\right] \qquad s = 3, 6, 15$$

Quatrit: moduli space

A quatrit 2-dimensional moduli space describe 2-parametric family of kernels that differ by their spectrum $\mathrm{spec}(\Delta) = \{\pi_1, \pi_2, \pi_3, \pi_4\}$.

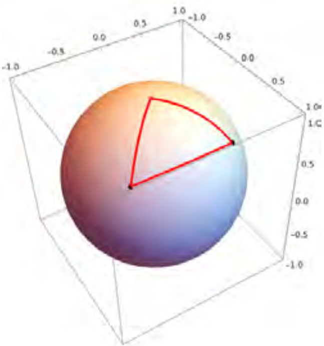

Figure 4: The Möbius spherical triangle as moduli space of WF of a quatrit.

Fixing the ordered spectrum of SW kernels of quatrit, $\pi_1 \geq \pi_2 \geq \pi_3 \geq \pi_4$ we identify the moduli space of quatrit with one out of 24 Möbius triangles $(2, 3, 3)$ which tessellate 2-sphere in the space of eigenvalues, parameterized by $\boldsymbol{\mu} = (\mu_3, \mu_6, \mu_{15})$:

$$\boldsymbol{\mu}^2 = 1\,,$$

$$\mu_3 \geq 0,\,, -\mu_6 \leq \mu_{15} \leq \mu_6$$

2-qubit; Local Unitary group

Below I consider the adjoint action $K = SU(2) \times SU(2) \subset SU(4)$ induced by a certain embedding of the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ into $\mathfrak{su}(4)$ via the compositions of two types of embedding:

$$\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(4) : \quad \begin{bmatrix} x & z \\ \bar{z} & -x \end{bmatrix} \hookrightarrow \left[\begin{array}{cc|cc} x & 0 & z & 0 \\ 0 & x & 0 & z \\ \hline \bar{z} & 0 & -x & 0 \\ 0 & \bar{z} & 0 & -x \end{array}\right]$$

and

$$\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(4) : \quad \begin{bmatrix} y & w \\ \bar{w} & -y \end{bmatrix} \hookrightarrow \left[\begin{array}{cc|cc} y & w & 0 & 0 \\ \bar{w} & -y & 0 & 0 \\ \hline 0 & 0 & y & w \\ 0 & 0 & \bar{w} & -y \end{array}\right]$$

The corresponding exponent mapping $\mathfrak{su}(4) \rightarrow SU(4)$ defines the embedding of the group $SU(2) \times SU(2)$ into $SU(4)$.

Double coset $SU(2) \times SU(2) \backslash SU(4) / T^3$ decomposition

Proposition: The $\mathfrak{su}(4)$ algebra admits decomposition into the direct sum:

$$\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{k}' ,$$

where $\mathfrak{a}$ and $\mathfrak{a}'$ are Abelian subalgebras such that

$$[\mathfrak{a}' , \mathfrak{a}] \subset \mathfrak{k} ,$$

and $\quad \mathfrak{k} := \mathfrak{su}(2) \oplus \mathfrak{su}(2) , \quad \mathfrak{k}' := \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) ,$ with the relations:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} , \qquad [\mathfrak{k}', \mathfrak{k}'] \subset \mathfrak{k}' , \qquad [\mathfrak{k}, \mathfrak{k}'] \subset \mathfrak{a} \oplus \mathfrak{a}' ,$$

The exponential map $\exp : \mathfrak{su}(4) \to SU(4)$ results in corresponding Cartan type coordinates description of the group in vicinity of the identity,

$$g := K \exp(\mathfrak{a}) \exp\left(\mathfrak{a}'\right) T_3 , \qquad K \in SU(2) \times SU(2) ,$$

where T_3 is the maximal torus in $SU(4)$,

Fano form of 2-qubit SW kernel

2-qubit SW kernel in Fano basis:

$$\Delta(\Omega_4) = \frac{1}{4}\,\mathbb{I}_4 + \frac{\sqrt{30}}{4}\left[\boldsymbol{\zeta_A}\cdot\boldsymbol{\sigma_A} + \boldsymbol{\zeta_B}\cdot\boldsymbol{\sigma_B} + \frac{1}{\sqrt{2}}\mathcal{E}_{ij}\,\sigma_i\otimes\sigma_j\right], \qquad (3)$$

where $\boldsymbol{\sigma_A} = \frac{1}{\sqrt{2}}(\sigma_{10}, \sigma_{20}, \sigma_{30})$, $\boldsymbol{\sigma_B} = \frac{1}{\sqrt{2}}(\sigma_{01}, \sigma_{02}, \sigma_{03})$. The coefficients of expansion $\boldsymbol{\zeta_A}$ and $\boldsymbol{\zeta_B}$ are a real 3-vectors and $\mathcal{E}$ is a real 3×3 matrix. According to the master equations for composite system the norm of these vectors and matrices is fixed:

$$\boldsymbol{\zeta_A}^2 = \frac{1}{5}, \qquad \boldsymbol{\zeta_B}^2 = \frac{1}{5}, \qquad \mathrm{tr}\left(\mathcal{E}\mathcal{E}^T\right) = \frac{3}{5}.$$

Hence, for 2-qubit all three primary second order $SU(2)\times SU(2)$ polynomial invariants of SW kernel are fixed, while the higher order invariants characterize all admissible types of SW kernels.

Subsystems kernels

Proposition: From SVD of SW kernel with the Cartan type coordinates for $SU(4)$ factor

$$\Delta(\boldsymbol{z}) = U(\boldsymbol{z})\mathrm{diag}(\pi_1, \pi_2, \ldots, \pi_N)U(\boldsymbol{z})^\dagger\,,$$

it follows that the reduced SW kernels of subsystems are:

$$\Delta_A = \frac{1}{2}\, U_A \left(\mathbb{I}_2 + \sqrt{15}\,(\zeta^A \cdot \boldsymbol{\sigma})\right) U_A^\dagger\,,$$

and

$$\Delta_B = \frac{1}{2}\, U_B \left(\mathbb{I}_2 + \sqrt{15}\,(\zeta^B \cdot \boldsymbol{\sigma})\right) U_B^\dagger\,,$$

with 3-vectors whose length is fixed by the “subsystem master equations"

$$(\zeta^A \cdot \zeta^A) = (\zeta^B \cdot \zeta^B) = \frac{1}{5}$$

The moduli space of 2-qubit

Proposition: The moduli parameters of 2 qubits are determined by the moduli parameters of SW kernel of 4-level system as whole μ_3, μ_6, μ_{15} and the lengths of 3-vectors

$$\zeta_i^A = \sum_{\alpha \in H} \mu_\alpha O_{\alpha i}\,, \qquad i = 1, 2, 3\,,$$

$$\zeta_i^B = \sum_{\alpha \in H} \mu_\alpha O_{\alpha i}\,, \qquad i = 4, 5, 6\,,$$

defined in terms of the component $\mathcal{A}$ the Cartan decomposition $K\mathcal{A}T_3$:

$$\mathcal{A}\lambda_\alpha \mathcal{A}^\dagger = O_{\alpha\beta}\,\lambda_\beta\,, \qquad \mathcal{A} = \exp\{\mathfrak{a}\}\exp\{\mathfrak{a}'\}$$

The moduli space of 2-qubit

The moduli space $\mathcal{P}_{2\times 2}$ of 2-qubits is given by the bundle of a unit 2-sphere, two ellipsoids $\mathrm{E_A}$, and $\mathrm{E_B}$ in the moduli space space $\mathcal{P}_4$ with coordinates $\boldsymbol{\mu} = \{\mu_3, \mu_6, \mu_{15}\}$:

$$\boldsymbol{\mu}\boldsymbol{\mu}^T = 1\,, \qquad \mathrm{E_A}: \quad \boldsymbol{\mu}\mathbb{A}\boldsymbol{\mu}^T = 1\,, \qquad \mathrm{E_B}: \quad \boldsymbol{\mu}\mathbb{B}\boldsymbol{\mu}^T = 1\,,$$

The 3×3 matrices $\mathbb{A}$ and $\mathbb{B}$ are:

$$\mathbb{A}_{\alpha\beta} := 5 \sum_{i=1,2,3} O_{\alpha i} O^T_{i\beta}\,, \qquad \mathbb{B}_{\alpha\beta} := 5 \sum_{i=4,5,6} O_{\alpha i} O^T_{i\beta}\,.$$

Properties of $\mathcal{P}_{2\times 2}$ are encoded in pairwise characteristic polynomials:

$$f_{\mathrm{E_A}\cap\mathbb{S}_2} = \det(t\mathbb{I}_3 + \mathbb{A}), \quad f_{\mathrm{E_B}\cap\mathbb{S}_2} = \det(t\mathbb{I}_3 + \mathbb{B}), \quad f_{\mathrm{E_A}\cap\mathrm{E_B}} = \det(t\mathbb{A} + \mathbb{B})\,.$$

Proposition: The ellipsoids and the 2-sphere overlap iff the characteristic polynomials $f_{\mathrm{E_A}\cap\mathbb{S}_2}, f_{\mathrm{E_B}\cap\mathbb{S}_2}$ and $f_{\mathrm{E_A}\cap\mathrm{E_B}}$ have no positive roots.

Experiment II: Searching for a "compositeness"

Algorithm of the generation of random SW kernels of 2-qubits

- Generate 4×4 unitary matrix U from the random Haar ensemble
- Compute $\mathbb{A}$ and $\mathbb{B}$ and find the spectrum $\boldsymbol{\mu}$ of SW kernel from:

$$\boldsymbol{\mu}\boldsymbol{\mu}^T 1\,, \qquad \boldsymbol{\mu}\mathbb{A}\boldsymbol{\mu}^T 1\,, \qquad \boldsymbol{\mu}\mathbb{B}\boldsymbol{\mu}^T = 1\,,$$

Probability of a “compositeness”

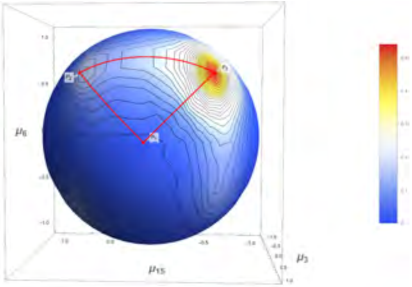

Figure 5: The probability distribution of 2-qubit SW kernels over the moduli space of quatrit.

1. Effect of the concentration of 2-qubits SW kernels around one of the vertex of Möbius triangle, $P_3 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
2. Probability of a “compositeness”:

$$\frac{\#\text{SW kernels of 2-qubits}}{\#\text{SW kernels of quatrits}} \approx 0.158$$

Beyond the talk

Topics around dichotomy "composite vs. elementary"

- Comparing nonclassicality indicators;
- Interrelations - positivity of WF & separability & entanglement ;
- Interrelations - marginals of WF and SW kernels .
- . . .

Instead of Conclusion

"Мне было очень отрадно установить, что квантовая механика лишает мир постного лика, который навязывает ему примитивный детерминизм. В свете этой науки весь мир предстает как азартная игра изобретального случая"

Д.И. Блохинцев
Мой путь в науке (автореферат работ)

"I found it very gratifying to establish that quantum mechanics deprives the world of the dreary face imposed on it by primitive determinism. In the light of this science, the whole world appears to be a gamble of an ingenious chance."

D.I.Blokhintsev
My Life in Science. (Summary of scientific works)