PI-type fully symmetric quadrature rules on the 3-, . . . , 6-simplexes

G. Chuluunbaatar[1,4,5], O. Chuluunbaatar[1,2,3], A.A. Gusev[1,4], S.I. Vinitsky[1,5]

[1]Joint Institute for Nuclear Research, Dubna, Moscow Region, Russia
[2]Mongolian Academy of Sciences, Ulaanbaatar, Mongolia
[3]Mongolian University of Science and Technology, Ulaanbaatar, Mongolia
[4]Dubna State University, Dubna, Russia
[5]Peoples' Friendship University of Russia (RUDN University), Moscow, Russia

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Outline

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- Solving system of nonlinear equations with convex constraints
- Conclusions

Fully symmetric quadrature rules for the d-simplex

p-order quadrature rule

$$\int_{\Delta_d} V(\boldsymbol{x}) d\boldsymbol{x} = \frac{1}{d!} \sum_{j=1}^{N_{dp}} w_j V(x_{j1}, \dots, x_{jd}), \tag{1}$$

$$\boldsymbol{x} = (x_1, \dots, x_d), \quad d\boldsymbol{x} = dx_1 \cdots dx_d,$$

Δ_d – the standard unit d-simplex;
N_{dp} – is the number of nodes
w_j – are the weights
$(x_{j1}, \dots, x_{jd})$ – are the nodes.

Barycentric coordinates $(y_1, \ldots, y_{d+1})$ nodes:

Following [a], the orbit $S_{[i]} \equiv S_{m_1 \cdots m_{r_{di}}}$:

[a] J.I. Maeztu, et al, Math. Comp. 64, 1171–1192 (1995).

$$(y_1, \ldots, y_{d+1}) = (\overbrace{\lambda_1, \ldots, \lambda_1}^{m_1 \text{ times}}, \ldots, \overbrace{\lambda_{m_{r_{di}}}, \ldots, \lambda_{m_{r_{di}}}}^{m_{r_{di}} \text{ times}}), \tag{2}$$

$$\sum_{j=1}^{r_{di}} m_j = d+1, \quad \sum_{j=1}^{r_{di}} m_j \lambda_j = 1, \quad m_1 \geq \cdots \geq m_{r_{di}},$$

$$P_{di} = \frac{(d+1)!}{m_1! \cdots m_{r_{di}}!} \quad \text{– the number of nodes of the orbit.}$$

System of nonlinear algebraic equations:

$$\int_{\Delta_d} s_2^{l_2} \times \cdots \times s_{d+1}^{l_{d+1}} d\boldsymbol{x} = \frac{1}{d!} \sum_{i=0}^{M_d} P_{di} \sum_{j=1}^{K_{di}} W_{i,j} s_{i,j2}^{l_2} \times \cdots \times s_{i,jd+1}^{l_{d+1}}, \tag{3}$$

$$2l_2 + \cdots + (d+1)l_{d+1} \leq p,$$

$$s_k = \sum_{l=1}^{d+1} x_l^k, \quad x_{d+1} = 1 - \sum_{i=1}^{d} x_i, \tag{4}$$

$$s_{i,jk} = \sum_{l=1}^{r_{di}} m_l \lambda_{i,jl}^k, \quad k = 2, \ldots, d+1.$$

Fully symmetric quadrature rules for the d-simplex

$$E_{dp} = \begin{cases} 1 + \left\lfloor \frac{p}{2} \right\rfloor, & d = 1, \quad p \geq 0 \\ E_{d-1p}, & d \geq 2, \quad 0 \leq p \leq d, \\ E_{d-1p} + E_{dp-d-1}, & d \geq 2, \quad p \geq d+1, \end{cases} \tag{5}$$

Table 1: The numbers E_{dp} of independent equations for fully symmetric p-order quadrature rules.

	E_{dp}				
p	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
4	4	5	5	5	5
6	7	9	10	11	11
8	10	15	18	20	21
10	14	23	30	35	38
12	19	34	47	58	65
14	24	47	70	90	105
16	30	64	101	136	164
18	37	84	141	199	248
20	44	108	192	282	364

Solving system of nonlinear equations with convex constraints

System of nonlinear equations with convex constraints

$$f_i(\boldsymbol{x}) = 0, \quad i = 1, \ldots, m, \quad \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{X}, \tag{6}$$

where $\mathcal{X} \subseteq R^n$ is a nonempty, closed and convex set.

The Levenberg-Marquardt-type algorithm

$$\min_{\boldsymbol{h} \in \mathcal{X}} G_k(\boldsymbol{h}), \quad G_k(\boldsymbol{h}) = \frac{1}{2}\|\boldsymbol{F}(\boldsymbol{x}^k) + \boldsymbol{J}_k \boldsymbol{h}\|^2 + \frac{1}{2}\mu_k(\boldsymbol{h}, \mathbf{D}_k \boldsymbol{h}), \tag{7}$$

where $\mathbf{J}_k \in R^{m\times n}$ is a Jacobian of $\boldsymbol{F}(\boldsymbol{x})$ at $\boldsymbol{x} = \boldsymbol{x}^k$, $\mathbf{D}_k \in R^{n\times n}$ is a positive diagonal matrix, and μ_k is a positive parameter.

Solving system of nonlinear equations with convex constraints

Table 2: The minimal numbers N_{dp} of nodes for PI-type fully symmetric p-order quadrature rules and comparison with the known numbers N_{dp}.

	N_{dp}								
p	$d=2$	$d=3$		$d=4$		$d=5$		$d=6$	
	cur., [1]	cur.	[2]	cur.	[3]	cur.	[4]	cur.	[6]
4	6	14	14	20	20	27	27	43	43
6	12	24	24	56	56	102	102	175	175
8	16	46	46	105	105	228	257	448	553
9	19	59	59	151	151	338		700	
10	25	79	81	210	210	479		1078	
12	33	123	168	370	445				
16	55	248	304	956	1055				
20	79	441	552						

[1] H. Xiao, et al, Comput. Math. App. 59, 663–676 (2010).
[2] J. Jaśkowiec, et al, Int. J. Numer. Methods Eng. 122, 148–171 (2021).
[3] C.V. Frontin, et al, App. Numer. Math. 166, 92–113 (2021).
[4] A.A. Gusev, ..., O. Chuluunbaatar, et al, LNCS 11077, 197–213 (2018).

Conclutions

Presented PI-type fully symmetric quadrature rules up to 20-th order on the tetrahedron, 16-th order on 4-simplex, 10-th order on 5- and 6-simplexes with almost minimal numbers of nodes.

The End

Thank you for attention!