ϕ^4 oscillons as standing waves in a ball: a numerical study

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MMCP'2024, Yerevan, Armenia, 20-26 Oct 2024

Equation

We consider the ϕ^4 equation

$$\Phi_{tt} - \Delta\Phi - \Phi + \Phi^3 = 0, \quad \Delta = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \tag{1}$$

which has a number of physical and mathematical applications.

Localized long-lived pulsating states (pulsons, oscillons) in the three-dimensional ϕ^4 theory are of special interest within a wide range of cosmological and high-energy physics contexts.

The earliest observations of repeated expansions and contractions of spherically-symmetric vacuum domains in the ϕ^4 equation were obtained in:

Voronov, Kobzarev, Konyukhova, JETP Lett **22** 290 (1975).

Simulations

Computer simulations revealed the formation of long-lived pulsating structures of large amplitude and nearly unchanging width

Bogolyubskii & Makhankov, JETP Lett **24** 12 (1976)
Bogolyubskii & Makhankov, JETP Lett **25** 107 (1977)

Example of numerical simulations of pulsating solution of Eq.(1)

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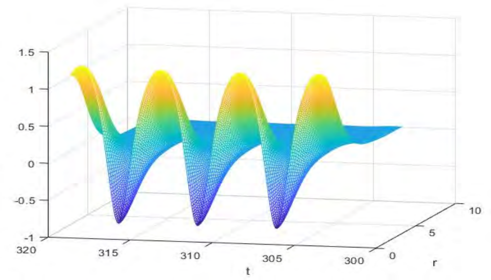


Aim & approach

- With their permanent loss of energy to the second-harmonic radiation, the oscillons are not exactly time-periodic.
- These infinite-space solutions can be studied via their approximation by standing waves in a ball of a finite radius.
- Unlike oscillons, the standing waves are exactly periodic and can be determined as solutions of a boundary-value problem on the cylindrical surface.
- Thus, our study aims an understanding of structure and properties of the oscillon by examining the periodic standing wave in a ball of finite radius R.

– N.Alexeeva, I.Barashenkov, A.Bogolubskaya, E.Zemlyanaya // Phys Rev **D** **107** (2023) 076023;
– E.Zemlyanayaa, A.Bogolubskayaa, M.Bashashin, N.Alexeeva. Phys. Part. Nucl. **55** No. 3 (2024) 505-508;
– E.Zemlyanaya, A.Bogolubskaya, N.Alexeeva M.Bashashin // Discrete & Contin. Models and Appl. Comput. Sci. **32** No. 1 (2024) 106-111

Boundary value problem

We consider the following boundary value problem:

$$\phi_{tt} - \phi_{rr} - \frac{2}{r}\phi_r + 2\phi - 3\phi^2 + \phi^3 = 0, \tag{2a}$$

$$\phi_r(0,t) = 0, \quad \phi(R,t) = 0, \quad \phi(r,T) = \phi(r,0). \tag{2b}$$

- Dependence of structure and properties of standing waves on the radius R and period T is numerically investigated.
- Numerical approach is based on numerical continuation and stability analysis of solutions of a 2D boundary value problem for the corresponding nonlinear PDE on the domain $[0,T]\times[0,R]$ where T – period of oscillations.
- Stability analysis is based on the Floquet theory.

Energy and frequency

The periodic standing waves are characterised by their energy

$$E = 4\pi \int_0^R \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2} + \phi^2 - \phi^3 + \frac{\phi^4}{4} \right) r^2 dr \qquad (3)$$

and frequency

$$\omega = \frac{2\pi}{T}. \qquad (4)$$

If the solution with frequency ω does not change appreciably as R is increased — in particular, if the energy (3) does not change – this standing wave provides a fairly accurate approximation for the periodic solution in an infinite space.

We analyse the boundary-value problem (2) and construct the $E(R)$ and the $E(\omega/\omega_0)$ dependence (where $\omega_0 = \sqrt{2}$).

Numerical approach

Letting $\tau = t/T$ and defining $\psi(r,\tau) = \phi(r,t)$ yields the boundary value problem at 2D domain $[0,1]\times[0,R]$:

$$\psi_{tt} + T^2 \cdot [-\psi_{rr} - \frac{2}{r}\psi_r + 2\psi - 3\psi^2 + \psi^3] = 0, \qquad (5a)$$

$$\psi_r(0,t) = \psi(R,t) = 0, \quad \psi(r,1) = \psi(r,0). \qquad (5b)$$

- Solutions of Eq.(5) were numerically continued in T and R to construct the energy diagram.
- For each values T and R the boundary-value problem (5) was solved by means of the Newtonian iteration with the 4th order finite difference approximation of the derivatives.
- Initial guess for the Newtonian process was calculated using the results at two previous continuation steps.

Stability analysis

To classify the stability of the resulting standing waves against spherically-symmetric perturbations we considered the linearised equation

$$y_{tt} - y_{rr} - \frac{2}{r} y_r - y + 3(\phi - 1)^2 y = 0 \qquad (6)$$

with the boundary conditions $y_r(0,t) = y(R,t) = 0$. We expand $y(r,t)$ in the sine Fourier series, substitute the expansion to Eq. (6) and, after transformations, finally obtain a system of $2N$ ODEs wrt unknown time-dependent Fourier coefficients:

$$\dot{u}_m = v_m, \qquad \dot{v}_m + \mathcal{F} = 0, \qquad (7)$$

$$\mathcal{F} = (2 + k_m^2) u_m - 3 \sum_{n=1}^{N} (A_{m-n} - A_{m+n}) u_n + \frac{3}{2} \sum_{n=1}^{N} (A_{m-n} - A_{m+n}) u_n,$$

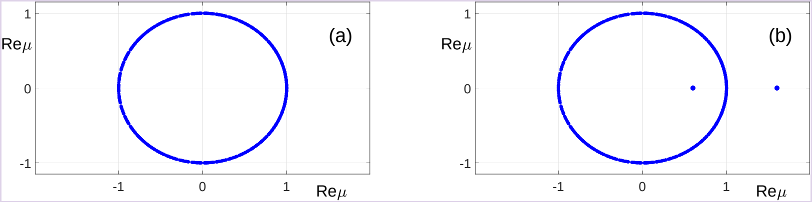
A_n, B_n are periodic functions of t, with period T:

$$A_n(t) = \frac{2}{R} \int_0^R \phi(r,t) \cos(k_n r) dr, \quad B_n(t) = \frac{2}{R} \int_0^R \phi^2(r,t) \cos(k_n r) dr$$

Calculation of Floquet multiplyers

The system (7) is solved, numerically, $2N$ times with series of varied initial conditions at the time-interval $[0, T]$ in order to form a matrix M_T. Eigenvalues $\mu = \exp(\lambda T)$ of M_T are the Floquet multipliers. The solution $\phi(r,t)$ is deemed stable if all its Floquet multipliers lie on the unit circle $|\zeta| = 1$ and unstable if there are multipliers outside the circle.

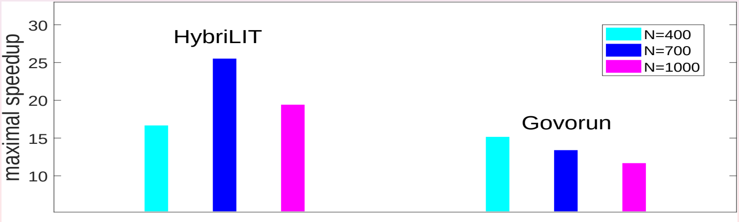
Floquet multipliers at the (Reμ,Imμ) plane. Stability case: T=4.7206, instability case: T=5.025. Here R=100.



Numerical approach, parallel implementation

Parallel MATLAB implementation:

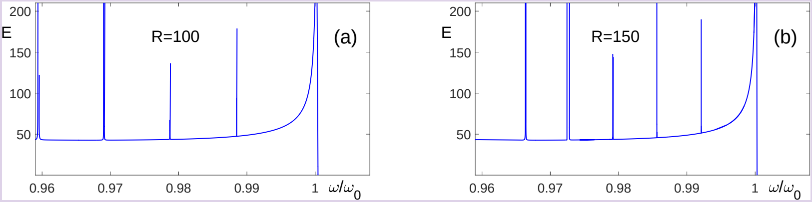
- The *ode45* procedure for numerical solution of the initial value problem (8) with the tolerance parameter value 10^{-7};
- Cubic spline interpolation for $A_{m\pm n}$ and $B_{m\pm n}$ coefficients for a set of time points.
- Operator *parfor* to provide parallel numerical solution of $2N$ Cauchy problems into available parallel threads, or "workers".



Energy-frequency diagram

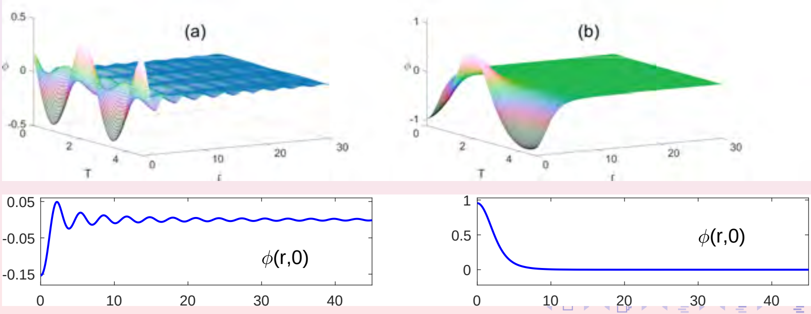
- The branch of ϕ comes from E=0 at Ω_1.
- Continuation produces curve $E(\omega/\omega_0)$ with a sequence of spikes; number and positions of spikes are R-sensitive.
- The lower envelope E-curve does not depend on R; it has a single minimum for all values of R, $\omega_{min}=\omega/\omega_0=0.967$, $E_{min}=42.74$.
- Stability occur only in case of frequencies lower ω_{min}.

Fragments of $E(\omega/\omega_0)$ diagram for $R=100$ and $R=150$:

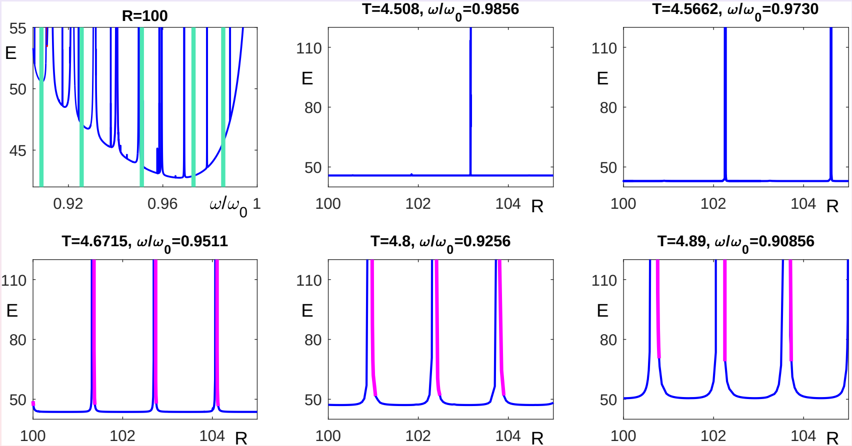


Two co-existing types of standing waves

1. Bessel-like waves without explicitly localized core, which are branching off the zero solution and decaying in proportion to r^{-1} as $r \to R$.
2. Nonlinear standing wave in a ball with an exponentially localised pulsating core and a small-amplitude slowly decaying second-harmonic tail.

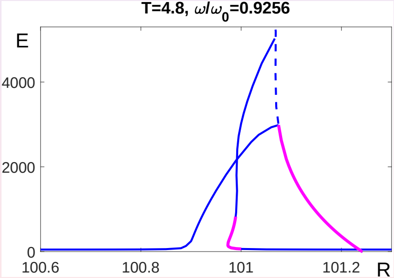


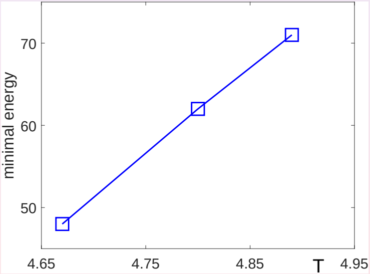
$E(R)$ diagram at several values of T: periodicity & stability properties



Interconnection of branches in case T=4.8 (left); Minimal E of stable waves vs T (right)

- blue solid: "standard" wave
- blue dashed: Bessel-like wave
- magenta: stable intervals





Summary

- R-periodicity of structure and stability properties of ϕ^4 standing waves is shown. Distance between E-peaks is T-dependent.
- Regions of stability on the $E(R)$ diagram are localized at the foot of the right slopes of the energy peaks.
- Both slopes of the $E(R)$ peak join the branch of Bessel-like waves at the period-doubling bifurcation points.
- Bessel-like waves are stable at the region between $E = 0$ and the period-doubling bifurcation point.
- One expects that for each $\omega/\omega_0 < \omega_{min}$, there is an equidistant sequence of R where the standing waves are stable.
- We obtained that minimal E at which the standing wave can be stable increases with decreasing frequency. This hypothesis needs to be checked at low frequencies.

The work is supported by the UCT/RSA – JINR Scientific Cooperation Program. We thank the HybriLIT team for the help with organization of calculations at the JINR Multifunctional Information Computing Center.

THANK YOU FOR YOUR ATTENTION!